Idiosyncratic Volatility, Firm Investment and Capital Accumulation

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August 31, 2018

Abstract

The goal of this paper is to quantify variation in the volatility of firm-level productivity shocks and study its impact via the accumulation of capital across firms. I first document robust empirical evidence on the upward trend in firm-level productivity shocks volatility. Then, I develop a tractable general equilibrium model to study the consequences of the increase in idiosyncratic volatility. The model features heterogeneous firms which make irreversible investment decisions over time. The volatility of idiosyncratic productivity shocks has impacts on investment mainly through two channels. The first one is the partial equilibrium real option effect. When the volatility of productivity shocks is high, the real option value of waiting increases and firms thus delay their investments. The second channel works through the general equilibrium effect of interest rates on investment. In equilibrium, the fall in aggregate investment corresponds to expected future decline in consumption growth and thus lower real interest rates. The decrease in interest rates would spur investment and thus counteract the partial equilibrium real option effect.

Keywords: Real options, investment, cross-sectional distribution, productivity

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1 Introduction

The dynamics at the firm-level and its relationship with aggregate economic fluctuations are important economic questions that receive expanding attentions. In particular, less is known about the dynamics of firm-level volatility and its impacts on aggregate economy.

This chapter first develops empirical methods to robustly quantify variations in the volatility of firm productivity shocks. I document empirical evidence that there exists an upward trend in the volatility of firm-level productivity shocks. The trend remains robust after controlling for composition change of the data sample. I argue that the upward trend in volatility is likely reflecting the fundamental change of the economy. Moreover, the upward trend in the volatility of firm-level productivity shocks is even stronger for firms which are younger, smaller and are in the technology sector. This finding contributes to the literature on firm-level risk. While Campbell et al. (2001) and Comin and Mulani (2006) discover the upward trend in the stock return and sales growth volatility, my paper provides evidence that the rise in the firm level risk may be driven by that in the volatility of productivity shocks.

Bloom et al. (2012) use the confidential Census Bureau data to measure the volatility of aggregate and idiosyncratic productivity shocks and find that both the aggregate and idiosyncratic volatility are countercyclical at the business cycle frequency. They measure the volatility of idiosyncratic productivity shocks as the cross-sectional dispersion of firm productivity shocks. Bachmann and Bayer (2013) also measures the volatility of firm productivity shocks using a Germany firm-level data and studies whether the volatility of firm level productivity shock is a major source of business cycle fluctuations. My empirical analysis complements Bloom et al. (2012) in several dimensions. First, I robustly quantify the dynamics of the idiosyncratic volatility, controlling for the composition change of data sample and other firm characteristics. Second, I accommodate the estimation of firm production function using the widely used Compustat database. In addition, I focus on the variation in idiosyncratic volatility at longer frequencies: the upward trend over the last five decades.

Existing papers, such as Bloom (2009) and Bachmann and Bayer (2013), have studied the short-run impact of idiosyncratic productivity shocks through dynamic models of firm investment. Their analysis focuses on the business cycle frequency variations. The main insight is that higher volatility increases the real option value of waiting and firms thus delay their investment. And the real option channel can operate through irreversibilities and non-convex adjustment costs of investment. Since these two factors qualitatively operates through the same real option channel, I focus on the irreversibility of investment for tractability reason. Another noteworthy paper in the investment literature is Khan and Thomas (2008). They study a model of lumpy investment caused by fixed adjustment cost wherein firms face shocks to the level of productivity. Khan and Thomas (2008) argues the general equilibrium effect of interest rates on investment is large enough to offset the partial equilibrium investment behaviors due to nonconvex adjustment costs. Even though my model studies the impact of changes to idiosyncratic volatility, it remains important to investigate through a general equilibrium model.

To quantitatively study the aggregate consequences of such changes on firm investment and capital accumulation, I build a tractable general equilibrium model with firm heterogeneity. In the model, there is an intertemporal optimizing representative consumer and a continuum of firms differing in their productivity. A crucial feature of the model is that investment is irreversible at the firm level. I find that the increase in idiosyncratic volatility has strong negative effect on the long-run investment and capital accumulation. Also, the short-run impact of the volatility of firm productivity shocks operates mainly through two channels. The first impact is the partial equilibrium real option effect, as in Bloom (2009). When the firm-level productivity shock volatility increases, the option value of waiting rises and firms delay their investments. The reallocation of capital to the most productive units thus stalls. However, the decrease in investments corresponds to expected future decline in consumption growth. The standard consumption Euler equation would predict lower real interest rates in equilibrium. The decrease in interest rates partially offset the partial equilibrium effect.

My model analysis contributes to the literature in a few ways. First, the model is based on the continuous-time firm investment model of Bertola and Caballero (1994). The key distinction is my model allows for more general dynamics for firm productivity and endogenizes aggregate interest rates in equilibrium. Second, Bloom (2009) and Bloom et al. (2012) only study the impact of shock idiosyncratic volatility at the business cycle frequency, I focus on both the short-run and long-run impact of changes in idiosyncratic volatility.

2 Empirical Results

The section develops measures for the volatility of firm-specific productivity shocks and explores potential causes for its variation over time.

2.1 Measuring Firm-specific Productivity Innovation

The benchmark proxy to capture the volatility of firm-specific innovations is the cross-sectional dispersion of future firm-specific productivity innovations. Following Bloom et al. (2012), idiosyncratic productivity is measured by firm-specific Solow residual. The log TFP innovations ($\epsilon_{i,t}$) are estimated based on the following first order autoregressive equation about log productivity ($\omega_{i,t}$).

$$\hat{\omega}_{i,t+1} = \rho_\omega \hat{\omega}_{i,t} + \mu_i + \lambda_{t+1} + \epsilon_{i,t+1} \tag{1}$$

where $\hat{\omega}_{i,t}$ denotes the estimated log TFP (Total Factor Productivity). The benchmark idiosyncratic volatility measure $\sigma_{\epsilon,t}$ is defined to be standard deviation of firm-specific TFP shocks $\epsilon_{i,t}$ across firms at a given time t.

The specification controls for the firm fixed effect: μ_i and the time fixed effect: λ_t. The log firm level TFP is estimated for a panel of firms using data from Compustat. The data spans annually from 1963 to 2015. The method of estimating firm-level productivity adopts from Olley and Pakes (1996), which has been used by Imrohoroglu and Tüzel (2014) recently. This semi-parametric method is advocated because it is able to control for simultaneity and selection bias. A selection problem is generated by the relationship between productivity and the shutdown decision, and a simultaneity problem is produced by the relationship between productivity and input demands. The details of this estimation method are provided in the appendix. Figure 1 plots the time-series of the volatility of firm-level productivity shocks. The underlying data frequency is annual.

2.2 The Dynamics of Idiosyncratic Volatility

In the previous section, I don't make any functional assumption on the time-series dynamics of the volatility of firm productivity shocks. The methods build on the assumption that productivity shocks across firms are independent of each other, even though the volatility of firm productivity shocks can vary over time. At each point in time, we have a large of sample of firm specific productivity shocks. Therefore, the cross-sectional dispersion of firm level productivity shocks is a valid estimator for firm specific shock volatility in the time dimension. Besides, when the number of firms in the cross section gets large as here using the Compustat dataset, the estimator becomes an accurate proxy for idiosyncratic volatility. Table 1 report the time-series properties of the estimator of the idiosyncratic volatility at the annual frequency. The idiosyncratic volatility is very persistent with an estimated AR(1) coefficient of 0.91 and standard deviation of 0.05. Therefore, I cannot reject the hypothesis that shocks to idiosyncratic volatility can be permanent. I also consider the properties of changes to idiosyncratic volatility. I find the changes have an AR(1) coefficient of only -0.002. Thus, changes to idiosyncratic volatilities are very persistent and therefore could have important economics implications.

2.3 An Alternative Way to Measure Idiosyncratic Volatility of Productivity Shocks

An relevant question is whether the change of idiosyncratic volatility of productivity shocks measured in Section 2.2 are due to changing characteristics of the data sample. One way to control for composition effect is to look at changes in the volatility of productivity shocks at the firm level. For a given firm i with data for date $t-1$, t, $t+1$. I use the standard cross-sectional regression approach Olley and Pakes (1996) to calculate the productivity residuals for firm i in year t and $t+1$. Squaring them and taking the difference produce a (very noisy) measure of the change in firm i's volatility of idiosyncratic productivity shocks from t to $t+1$. Let $\Delta Vol_{i,t+1} \equiv \epsilon^2_{i,t+1} - \epsilon^2_{i,t}$ denotes this change. For each date $t+1$, I calculate $\Delta Vol_{t+1,EW}$: the equal-weighted mean of the change of volatility across all firms with non-missing $\Delta Vol_{i,t+1}$ or $\Delta Vol_{t+1,VW}$: value-weighted mean using market equity value at time t. An advantage of using value-weighted mean is that it would prevent the estimates from biased towards productivity shocks of a large number of small firms. The last step is to keep track of the level of productivity from the change of volatility over time. Let $Vol_{t,EW} \equiv \sum_{s=1}^{t} Vol_{s,EW}$ and $Vol_{t,VW} \equiv \sum_{s=1}^{t} Vol_{s,VW}$ denote these measures for the level of idiosyncratic volatility.

Figure 2 plots the time-series of these estimates from 1963 to 2015. It is clear from Figure 1 and 2 that there exists a robust upward trend for the level of idiosyncratic volatility. The value-weighted measure of idiosyncratic volatility is in general smaller than the equal-weighted one. The reason is likely that weighting by market capitalization downplays the increase in idiosyncratic volatility of small firms. Both the equal-weighted and value-weighted measures are strongly correlated with the cross-sectional dispersion measure $\sigma_{\epsilon,t}$ defined in Equation (1). The equal weighted measure has a correlation coefficient of 0.90 with respect to $\sigma_{\epsilon,t}$, while the value-weighted measure is significantly correlated with $\sigma_{\epsilon,t}$ with a coefficient of 0.78. Therefore, it is plausible to assert that changes in idiosyncratic volatility are not driven by the composition change of the Compustat data sample.

2.4 Rolling Window Measure of Idiosyncratic Volatility

Another way of measuring the volatility inherent in the firm's environment is by focusing on the time series. Formally, I consider the rolling time series for the volatility of $\epsilon_{i,t}$ as

$$Vol_{i,R} = \sqrt{\frac{\sum_{\tau=t-9}^{t} (\epsilon_\tau - \bar{\epsilon}_t)^2}{10}} \tag{2}$$

where $\bar{\epsilon}_t \equiv \sum_{\tau=t-9}^{t} \epsilon_\tau$. This measure could be more appealing in that it is less likely to be affected by the composition effect. When computing the standard deviation in the times series, I remove the average growth rate for the firm in the window, and in effect control for firm-specific aspects that affect the growth rate of productivity. These aspects, however, potentially show up in the cross-sectional measure and may be the medium through which a compositional bias operates. These standard deviations are then averaged across all the firms in a year to arrive a the average volatility for every year. As illustrated in Figure 3, volatility at the firm level exhibits a significant upward trend. In order to build a more representative measure, the standard deviations are weighted using the firm's market equity in a given year. Even though the trend become flatter using the value weighted measure, the volatility of firm productivity has increased more than 100% from the early 1960s to 2000s.

2.5 Controlling for Firm Size, Age and Sectors

I have presented three different measures for idiosyncratic productivity shocks volatility. Since the time series of all measures display the same upward trend, I focus on the first measure: the cross-sectional dispersion $\sigma_{\epsilon,t}$ hereafter. Figure 4 exhibits the time-series of the volatility for firms with different sizes. In each year, I divide firms into three groups based on their market capitalization. We can see that the upward trend in volatility holds for firms in different size groups. Relatively, the trend increase is stronger for small firms and weaker for large firms.

I conduct a similar exercise for firms in different age groups. Figure 5 shows that younger firms have a stronger trend increase in the productivity shocks volatility, while older firms go through a relatively smaller increase. The volatility for younger firms increases from 0.1 in 1960s to more than 0.4 in early 2000s. Older firms witnessed a comparatively milder increase in volatility, which rises from 0.1 in 1960s to more than 0.3 in early 2000s.

I also examine four main industries in this paper: consumer goods, manufacturing, health products and information, computer and technology industries. The classification of consumer goods, manufacturing, and health products industries are taken from Fama-French 5-industry classification.[1] The information, computer and technology industry classification is from the BEA Industry Economic Accounts, which consists of computer and electronics products, publishing industries (including software), information and data processing services, and computer systems design and related services. The patterns in the Figure 6 are intriguing. The information technology sector witnesses the strongest increase in productivity shock volatility. The peak volatility is 0.54 in year 2001 while the highest volatility for across all firms is 0.38 at the same year. The consumer goods sector takes the smallest increase in volatility with the peak of 0.28 of in year 2012. Therefore, the dynamics of productivity shocks volatility exhibit a significant degree of heterogeneity. The increase of

[1] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/changes_ind.html

volatility is stronger for firms which are younger, smaller and making more investments, such as firms in the technology sector.

2.6 The Rise of Idiosyncratic Volatility

A striking pattern in previous figures is the substantial and robust rise of the volatility of firm-level productivity shocks over the last fifty years.This finding contributes to the literature on firm-level volatility dynamics. Campbell et al. (2001) discovers the upward trend in the cross-sectional dispersion of the component of returns that is unrelated to the average return in the four-digit sector. Comin and Mulani (2006) documents the same upward trend in firm sales growth volatility. My paper establishes that the volatility of productivity shocks has risen substantially, which suggests that the upward trend of firm risk may be firm fundamental driven. While Bloom et al. (2012) also measures the volatility of TFP shocks, their approach is mostly suited for the confidential Census Bureau data. My empirical approach is based on the Compusutat/CRSP dataset, which is more available for researchers. Besides, I conduct further robustness control and estimate the production functions. Therefore, my empirical analysis can be viewed as a complement to theirs.

I have reported idiosyncratic productivity volatility for firms with different size, age and sectors. Younger and smaller firms experience larger hikes in the volatility of productivity shocks. Firms in the information, computer and technology industry experience the largest increase in productivity shocks volatility, while consumer goods sector firms experience a relatively mild growth. Even though firms vary in the degree of growth in productivity shock volatility, the upward increase in volatility is robustly significant across firms. This highlights the potential aggregate consequences of such dynamics of volatility on aggregate investment and capital accumulation. To further quantitatively shed light on this question, it is therefore important to investigate through an economic model with emphasis on firm investment and volatility dynamics.

3 The Model

In this section, I analyze the quantitative impact of variation in idiosyncratic volatility within a continuous time dynamic general equilibrium model. Specifically, I consider an economy with heterogeneous firms that make irreversible investments to produce a final good. The economy consists of a representative household and a continuum of firms with a unit mass. Assume that there is no aggregate uncertainty and that firms face idiosyncratic productivity shocks. By the law of larger numbers, all aggregate quantities are deterministic over time, although each firm is still exposed to idiosyncratic uncertainty. Firms that adjust their capital stock incur adjustment costs.

To study the long-run impact of idiosyncratic volatility variation, I investigate the steady states properties of the model under different levels of idiosyncratic volatility. In particular, I focus on investment and capital accumulation. I also examine the short-run dynamics of the model through the transition path of the economy from one steady state to a new one, which is caused by an unexpected increase in the idiosyncratic productivity shock volatility. [2] [3]

3.1 Households

The economy is populated by a continuum of identical households. They have preferences with the same discount rate ρ, and elasticity of intertemporal substitution (EIS) ψ^{-1}. They are defined following, for example, Cass (1965)

$$U_t = \mathbb{E}_t \left[\int_t^{\infty} f(C_u) du \right], \quad f(C_u) = \rho \frac{C^{1-\psi}}{1-\psi} \tag{3}$$

[2] Compared with a fully dynamic model in which the idiosyncratic volatility follows a stationary process and agents know that the volatility of idiosyncratic volatility is time-varying, my model would tend to over-predict the effect of an increase in the idiosyncratic volatility. In my setup, agents don't have precautionary motives against changes of volatility and assume that the change of volatility is permanent.

[3] It is well known from papers such as Krusell and Smith (1998) and Khan and Thomas (2008) that the cross-sectional distribution of firm capital accumulation becomes infinite dimension state variables to keep track of in heterogeneous agents model with aggregate shocks. The absence of aggregate shocks in this paper significantly simplifies the state space and the solution of the model.

The term C_t is the consumption rate at time t. Let W denotes the wealth of the representative agent and $J(W)$ the value function. In equilibrium, it must be the case that $J(W_t) = U_t$. To solve for the household value function, consider the Hamilton-Jacobi-Bellman (HJB) equation for an investor who allocates wealth between the claim to all dividends (stock market) in the economy and the risk-free asset. Since there is no aggregate risk in the economy, wealth follows the process

$$dW_t = (r_t W_t - C_t)dt$$

The solution to the representative agent's consumption and portfolio choice problem is given by the following HJB equation (Duffie and Epstein (1992))

$$0 = \max_{C_t} f(C_t) + J_W[r_t W - C_t] \tag{4}$$

Taking the first-order condition with respect to C, we have

$$f_C(C_t) - J_W = C_t^{-\psi} - J_W = 0$$

For further analysis, it is convenient to calculate the state-price density, which prices consumption goods in different states of the world. In particular, it can be shown (e.g., Duffie and Epstein (1992)) that the state-price density Λ_t equals to

$$\Lambda_t = \exp(-\rho t) C_t^{-\psi} \tag{5}$$

By the law of one price, the state price density evolves following

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt$$

where r_t is the real risk-free interest rate in the economy. Since there is no aggregate uncertainty in the model, aggregate variables including the risk-free rate are deterministic over time. The interest rates could vary when the economy is making transitions to new steady states.

3.2 Firms

There is a continuum of firms indexed by $i \in [0,1]$ with idiosyncratic productivity $Z_{i,t}$. Productivity $Z_{i,t}$ follows some Markov process. The productivity of each firm is independent of each other. Each firm produces goods with productivity $Z_{i,t}$. This standard setup for firm heterogeneity can be seen, for example, at Khan and Thomas (2008). The firm-level productivity follows a diffusion process

$$Z_{i,t} = \mu(Z_{i,t})dt + \sigma_t dW_{i,t} \tag{6}$$

It is important to note that the firm level productivity shock volatility can only change deterministically over time. This assumption is only made for tractability reason.

Let K_t denotes the firm's capital stock and the process L_t represents the cumulative gross investment up to date t. Investment is often irreversible in that installed capital has little or no value unless used in production. Following Bertola and Caballero (1994), I assume that investment at the firm level is irreversible in the sense that the capital has no resale value. Thus, it is never worthwhile for firms to disinvest and the gross investment process L_t is nondecreasing over time. Ramey and Shapiro (2001) suggests that this assumption is realistic, at least for some industries. Capital depreciates at the constant rate $\delta \geq 0$, so the stochastic process for the capital stock of firm i is

$$dK_{i,t} = dL_{i,t} - \delta K_{i,t}dt \tag{7}$$

Summing over all firms in the economy, the law of motion for the aggregate capital stock is

$$dK_t = \int_i dK_{i,t} - \delta K_t dt$$

The firm's objective is to maximize expected discounted profits. Hence its problem is [4]

$$V(K_t, Z_t) = \max_{L(t+u),u\geq 0} \mathbb{E}_t \left[\int_0^\infty \frac{\Lambda_{t+u}}{\Lambda_t} \{\Pi(K_{t+u}, Z_{t+u})du - dL(t+u)\} \right] \tag{8}$$

[4] I suppress the firm specific subscript i for simplicity.

Under standard techniques (see e.g., Stokey (2008)), it is possible to show that the optimal investment policy is defined by a threshold function $b(Z)$. If $K < b(Z)$ the firm makes discrete investment of size $b(Z) - K$, so below the threshold the value function is

$$V(K, Z) = V((b(Z), Z)) + b(Z) - K, \quad K < b(Z) \tag{9}$$

The region above $b(Z)$ is the inaction region. In this region, the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem (10)

$$\begin{aligned} r_t V(K_{i,t}, Z_{i,t}) = [\Pi(K_{i,t}, Z_{i,t}) - \delta K_{i,t} V_K(K_{i,t}, Z_{i,t})] + \mu(Z_{i,t}) V_Z(K_{i,t}, Z_{i,t}) \\ + \frac{1}{2}\sigma(Z_{i,t})^2 V_{ZZ} \quad \text{if} \quad K \geq b(Z) \end{aligned} \tag{10}$$

The term on the left side of (10) denotes the expected interest of investing at time t. The first term on the right hand side gives the expected cash flow. The second term on the right gives the drift and volatility effects of productivity change on $V(K, Z)$.

3.3 Heterogeneity and Aggregation

In order to solve for the equilibrium, it is necessary to keep track of the cross-sectional distribution of firm capital stock to characterize the dynamics of the aggregate state of the economy. Firms in the economy are indexed by their productivity types Z and capital stock K. At each point in time t, it is important to keep track of the joint distribution of capital and productivity: $g_t(K, Z)$. The corresponding marginal distributions are denoted by $\phi_t(K)$ and $\psi_t(Z)$. It is straightforward to define the aggregate capital stock K_t as

$$K_t = \int_K \int_Z K g_t(K, Z) dK dZ \tag{11}$$

The total output in the economy Π_t then follows

$$\Pi_t = \int_K \int_Z K^\alpha Z^{1-\alpha} g_t(K, Z) dK dZ \tag{12}$$

Similarly, the aggregate investment I_t is represented by

$$I_t = \int_{K<b(Z)} \int_Z (b(Z) - K) g_t(K, Z) dK dZ \tag{13}$$

The next proposition is the main tool to characterize the evolution of the cross-sectional distribution of and capital stock K and productivity Z.

Proposition 1. ***(The Evolution of Cross-Sectional Distribution)*** *The cross-sectional distribution $g_t(K, Z)$ obeys the second order partial differential equation*

$$\partial_t g_t(K, Z) = \partial_K \left(\max(b(Z) - K, 0) g_t(K, Z)\right) - \partial_Z \left(\mu(Z) g_t(K, Z)\right) + \frac{1}{2}\partial_{ZZ} \left(\sigma^2(Z) g_t(K, Z)\right) \tag{14}$$

The partial differential equation is mathematically similar to the Kolmogorov Forward equation to keep track of the distributions of diffusion processes. This method to keep track of the cross-sectional distribution of firm capital stock is based on Achdou et al. (2014). While they study the heterogeneity on the household side, I focus on the heterogeneity at the firm level. [5]

3.4 Equilibrium

With the characterization of the optimal firm policies and aggregate quantities complete, I now state the definition of the competitive general equilibrium.

Definition 1. ***(Competitive Equilibrium)*** *A competitive equilibrium is a set of processes: aggregate consumption C_t, the state price density π_t, aggregate capital stock K_t; and a set of stochastic processes for each firm $i \in \mathbb{I}$: investment I_i, capital stock $K_{i,t}$, output $\Pi_{i,t}$ such that*

(1) The representative consumer and each firm solve their problems taking aggregate conditions as given.

[5] There are unfortunately no easy explanations for the proposition. An illustrative example is when Z is a constant. The last two terms on the RHS become zero. The first term on the RHS keeps track of the rate of change of K.

(2) Market clearing:

$$\int_i K_{i,t} di = K_t$$
$$\int_i \Pi_{i,t} di = \Pi_t$$
$$C_t + I_t = \Pi_t$$

(3) Aggregate capital stock satisfies the law of motion, starting from K_0:

$$dK_t = \int_i I_{i,t} dt - \delta K_t$$

The market clearing conditions for the consumption goods and capital market are standard. An important class of the equilibrium is the steady state of the economy which is defined as follows.

Definition 2. ***(Steady State)*** *The steady state of the economy is characterized by a competitive equilibrium path in which*
(1) The aggregate consumption growth rate and the risk-free rate r_t are constant over time.
(2) the cross-sectional distribution of firm capital stock is invariant over time.

In the steady state of the model, the stationary cross-sectional distribution of endogenous state variables has been reached. By the law of large numbers, economic aggregates are constants over time.

I now state the exact formulation of the equilibrium conditions

Proposition 2. ***(Equilibrium Conditions)*** *The equilibrium is characterized by the fol-*

lowing partial differential equation systems

$$
\begin{aligned}
0 =& \max_{C_t} f(C_t, J(W_t)) + J_W[r_t W_t - C_t] && (15)\\
r_t V(K_{i,t}, Z_{i,t}) =& [\pi(K_{i,t}, Z_{i,t}) - \delta K_{i,t} V_K(K_{i,t}, Z_{i,t})] + \mu(Z_{i,t}) V_Z(K_{i,t}, Z_{i,t}) \\
& + \frac{1}{2}\sigma(Z_{i,t})^2 V_{ZZ} \qquad \textit{if} \quad K \geq b(Z) && (16)\\
V(K, Z) =& V((b(Z), Z)) + b(Z) - K \quad \textit{if} \quad K < b(Z) && (17)\\
C_t + I_t =& \Pi_t && (18)\\
\partial_t g_t(K, Z) =& \partial_K \left(\max(b(Z) - K, 0) g_t(K, Z)\right) - \partial_Z \left(\mu(Z) g_t(K, Z)\right) + \frac{1}{2}\partial_{ZZ}\left(\sigma^2(Z) g_t(K, Z)\right) && (19)
\end{aligned}
$$

4 Computing Algorithms

The section develops the numerical solution for the heterogeneous firms model using a finite difference method of partial differential equations.[6] The finite difference methods have been successfully used to value options and other derivative securities. To obtain valuation formula for warrants, Schwartz (1977) proposes to use the finite difference method to numerically solve the partial differential equations. Hull and White (1990) extends the standard finite difference method to price a wider class of derivative securities. The flexibility of the finite difference method facilitates the computation of my model equilibrium.

4.1 Computing Steady States

In this section, I describe how I calculate the stationary equilibria. Since the aggregate economy is stationary and there exists a representative agent, the stationary interest can be proven to be equal to ρ: the time preference rate.

1. Given the steady state interest rate $r = \rho$, solve the firm's HJB equation (10) using a finite difference method. Calculate the investment threshold $b(Z)$.

[6] The appendix establishes analytical solutions under specific functional assumptions about the productivity process. The analytical solutions are helpful to develop intuitions and verify numerical results of the model.

2. Given the investment threshold function $b(Z)$, solve the Kolmogorov Forward equation (14) using a finite difference method.

3. Given the cross-sectional distribution of $g(K, Z)$ and investment threshold $b(Z)$, compute the aggregate output Π_t and aggregate investment I_t. The aggregate consumption is then $C_t = \Pi_t - I_t$ by the market clearing condition.

4. Given the aggregate consumption C, solve the HJB equation of the representative household. Compute J_W and r and verify that $r = \rho$

4.2 Finite Difference Methods

For step 1, the finite difference method approximates the functions V at I grid points in capital K, $K_i, i = 1, ...I$ and J grid points in productivity dimension Z, Z_j, $j = 1, ...J$. I use equispaced grids denote by ΔK and ΔZ the distance between grid points, and use short-hand notation $V_{i,j} = V(K_i, Z_j)$. The derivative $\partial_K V_{i,j} = \partial_K V(K_i, Z_j)$ is approximated with either a forward or backward difference approximation

$$\partial_{K,F} V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{\Delta K} \tag{20}$$

$$\partial_{K,B} V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{\Delta K} \tag{21}$$

Similarly, the finite difference for the derivation in productivity Z follows

$$\partial_Z V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\Delta Z} \tag{22}$$

$$\partial_{ZZ} V_{i,j} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta Z)^2} \tag{23}$$

Let n denotes the number of iterations implemented to find the solution to the HJB equation. I use the following finite difference approximation to updates the value function $V^n(K, Z)$ (10)

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = \Pi_{i,j}^n - \delta K_i \partial_K V_{i,j}^{n+1} + \mu_j \partial_Z V_{i,j}^{n+1} + \frac{\sigma_j^2}{2} \partial_{ZZ} V_{i,j}^{n+1} \tag{24}$$

where the parameter Δ is the step size.

4.3 Upwind Scheme

The upwind scheme is to use the forward approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative. Since in the inaction region, the capital stock size is drifting down due to depreciation. I use the following finite difference approximation

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1} = \Pi_{i,j}^{n}-\delta K_i\partial_{K,B}V_{i,j}^{n+1}+\mu_j\partial_Z V_{i,j}^{n+1}+\frac{\sigma_j^2}{2}\partial_{ZZ}V_{i,j}^{n+1} \tag{25}$$

Substituting the definitions for finite differences above, we have

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1} = \Pi_{i,j}^{n}-\delta K_i\frac{V_{i,j}^{n+1}-V_{i-1,j}^{n+1}}{\Delta K}+\mu_j\frac{V_{i,j+1}^{n+1}-V_{i,j}^{n}}{\Delta Z}+\frac{\sigma_j^2}{2}\frac{V_{i,j+1}^{n+1}-2V_{i,j}^{n+1}+V_{i,j-1}^{n+1}}{(\Delta Z)^2} \tag{26}$$

The equation (26) constitutes a system of $I\times J$ linear equations and can be written in matrix forms in the following steps. Collecting terms with the same subscripts on the right hand side, we have

$$\begin{aligned}
&\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1} = \Pi_{i,j}^{n}+x_{i,j}V_{i-1,j}^{n+1}+y_{i,j}V_{i,j}^{n+1}+\chi_j V_{i,j-1}^{n+1}+\zeta_j V_{i,j+1}^{n+1}\\
&x_{i,j}=\frac{\delta K_i}{\Delta K},\quad y_{i,j}=-x_{i,j},\quad \nu_j=-\frac{\mu_j}{\Delta Z}-\frac{\sigma_j^2}{(\Delta Z)^2}\\
&\chi_j=\frac{\sigma_j^2}{2(\Delta Z)^2},\quad \zeta_j=\frac{\mu_j}{\Delta Z}+\frac{\sigma_j^2}{2(\Delta Z)^2}
\end{aligned} \tag{27}$$

It is important to note that $x_{1,j}=0$ for all j because the size of the capital stock is bounded in the approximation schemes. At the boundaries of the productivity dimension j, the equation become

$$\frac{V_{i,1}^{n+1}-V_{i,1}^{n}}{\Delta}+\rho V_{i,1}^{n+1} = \Pi_{i,1}^{n}+x_{i,1}V_{i-1,1}^{n+1}+(y_{i,1}+\nu_1)V_{i,1}^{n+1}+\chi_1 V_{i,1}^{n+1}+\zeta_1 V_{i,2}^{n+1} \tag{28}$$

$$\frac{V_{i,J}^{n+1}-V_{i,J}^{n}}{\Delta}+\rho V_{i,J}^{n+1} = \Pi_{i,J}^{n}+x_{i,J}V_{i-1,J}^{n+1}+(y_{i,J}+\nu_J)V_{i,J}^{n+1}+\chi_J V_{i,J-1}^{n+1}+\zeta_J V_{i,J}^{n+1} \tag{29}$$

where I have used that $V_{i,0} = V_{i,1}$ and $V_{i,J} = V_{i,J+1}$. The equation (27) can be written in matrix notation as:

$$\frac{1}{\Delta}(V^{n+1} - V^n) + \rho V^{n+1} = \Pi^n + \mathbb{A}^n V^{n+1} \tag{30}$$

where V^n is a vector of length $I \times J$ with entries $(V_{1,1}, ..., V_{I,1}, ...V_{1,J}, ...V_{I,J})'$ and $\mathbb{A}^n = \mathbb{B}^n + \mathbb{C}$ where the $(I \times J) \times (I \times J)$ matrices $\mathbb{B}^n$ and $\mathbb{C}$ are defined as

$$\mathbb{B}^n = \begin{bmatrix}
y_{1,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
x_{2,1} & y_{2,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & x_{I,1} & y_{I,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & y_{1,2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & x_{2,2} & y_{2,2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{I,2} & y_{I,2} & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & y_{1,J} & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{2,J} & y_{2,J} & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & x_{I,J} & y_{I,J}
\end{bmatrix} \tag{31}$$

$$
\mathbb{C} = \begin{bmatrix}
\nu_1+\chi_1 & 0 & \cdots & \cdots & \zeta_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \nu_1+\chi_1 & 0 & \ddots & \ddots & \zeta_1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \nu_1+\chi_1 & 0 & \ddots & \ddots & \zeta_1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\chi_2 & \ddots & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \ddots & \ddots & \ddots & \vdots \\
0 & \chi_2 & \ddots & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & \chi_2 & 0 & \ddots & 0 & \nu_2 & 0 & \ddots & \ddots & \zeta 2 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & \ddots & \nu_J+\chi_J & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \chi_J & 0 & \ddots & 0 & \nu_J+\chi_J & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \chi_J & 0 & \cdots & 0 & \nu_J+\chi_J
\end{bmatrix} \tag{32}
$$

4.4 Kolmogorov Forward Equation

I now turn to the solution of the (19). The equation is discretized similar to the finite difference method used for the HJB equation. The technical details can be seen at Achdou et al. (2014).

4.5 Computing Transition Dynamics

I compute the transition dynamics using the following algorithm. Approximate the value function at N discrete points in the time dimension. Use the short-hand notation $v_{i,j}^n = V(K_i, Z_j, t^n))$. Guess a function $r^0(t)$, then for $\ell = 0, 1, 2..$ follow

(1) Given $r^\ell(t)$, solve the firm's HJB equation (10) with terminal condition $V^T(K, Z) = V(K, Z)$ backward in time to compute the time path of $V_{i,j}^n$. Also compute the implied investment threshold $b_t^\ell(Z)$.

(2) Given the investment threshold $b_t^\ell(Z)$, solve the Kolmogorov Forward equation with

initial condition $g_0^t(K,Z) = g(K,Z)$ forward in time to compute the time path for $g^\ell(K,Z,t)$.

(3) Given $g^\ell(K,Z,t)$ and $b_t^\ell(Z)$, calculate aggregate investment I_t and output Π_t.

(4) Given $r^\ell(t)$, solve the representative's agent's HJB equation (4) with terminal condition $J^T(W_T)$. Compute the consumption C_ℓ^n.

(5) Given $b_t^\ell(Z)$, $g^\ell(K,Z,t)$ and C_ℓ^n, calculate the surplus

$$S^\ell(t) = \Pi_t - C_t - I_t$$

.

(6) Update $r^{\ell+1}(t) = r^\ell(t) - \xi\frac{dS^\ell(t)}{dt}$, where $\xi > 0$.

(6) Stop when $r^{\ell+1}$ is sufficiently close to $r^\ell(t)$.

5 Quantitative Implications of Idiosyncratic Volatility

The main purpose of this section is to illustrate the impact of idiosyncratic volatility on aggregate investment and capital allocation both in steady states and during transitions. I show that idiosyncratic volatility has important implications for the quantities in steady states and transition dynamics. Bloom et al. (2012) structurally estimate a dynamic general equilibrium model to study the impact of uncertainty shocks at the business cycle frequency. My paper differs from theirs in the focus on idiosyncratic volatility and long-run capital accumulation.

5.1 The Productivity Process

The framework laid out in the previous section works with a relatively general process. In the numerical analysis of this section, I consider the case that idiosyncratic productivities are following Ornstein-Uhlenbeck processes.

$$dZ_t = \theta(\mu - Z_t)dt + \sigma dW_t \tag{33}$$

where μ represents the mean value of productivity, σ is the degree of volatility and θ the rate by which these shocks dissipate and the productivity reverts towards the mean. An attractive feature of this process is that it is the exact continuous time counterpart to a discrete-time AR(1) process. Also I specify the profit function $\Pi(K_t, Z_t) = K_t^{\alpha} Z_t^{1-\alpha}$ to be of constant returns to scale in (K_t, Z_t).

As a brief aside, I would like to note that for alternatives to the Ornstein-Uhlenbeck process, the steady state cross-sectional distribution of investment rate may be actually solved in closed forms. I provide one example in which productivity follows geometric Brownian motion in the Appendix. In that case, there is a strictly negative relationship between the idiosyncratic volatility of productivity shocks and firm investment.

5.2 The Parameters of the Model

In the model, time is continuous and the length of unit interval corresponds to one year. I set α: the capital share in the production function to 0.33. The persistence parameter of productivity: θ equals 0.3, which corresponds to an annual first-order autocorrelation of productivity of 0.7. The depreciation rate is set to 0.1 in the annual sense. The time preference rate ρ equals to 0.05. These values are standard in the macroeconomic literature. The choice of intertemporal elasticity of substitution (IES) is subject to discretion. Hall (1988) estimates the IES to be well below 1. Bansal and Yaron (2004) argues that an IES of 2 is important to reconcile asset pricing moments. I choose the inverse elasticity of intertemporal substitution to be 0.5 and 5 in my quantitative exercises.

In my quantitative exercises, I analyze steady states of the model with the volatility of productivity shocks $\sigma = 0.1$ and $\sigma = 0.2$. I also consider the transition path from the low volatility steady state to the high volatility one. These two values roughly correspond to the volatility of firm productivity shock in early 1960s and the more recent value. The values of the parameters are listed in Table 2. Since there are no aggregate shocks ex-ante, the aggregate consumption, investment and real interest rates are constants in the economy. The intertemporal elasticity of substitution plays no role in determining quantities in steady states.

5.3 The Effect of Idiosyncratic Volatility on Steady States

When there is no irreversibility constraint, the firm would always invest(disinvest) until the marginal value of capital equals to the price of capital. Since the investment adjustment cost is linear in the amount of capital invested, capital stock adjustment takes place immediately.[7] However, the irreversibility constraint here prevents the firm from adjusting its capital stock down if the current level of capital is larger than the optimal level of capital stock. Therefore, the investment policy in this model is characterized by the investment threshold $b(Z)$. At the threshold, the marginal value of capital equals to the price of capital. A firm expands its capital stock immediately to the threshold if its capital stock is lower than that. But the firm cannot downsize the capital stock if it is higher than the threshold.

Figure 7 graphs the investment threshold against firm productivity for two levels of idiosyncratic volatility. Two observations can be made. An increase of idiosyncratic productivity volatility has significant negative effect on the level of investment threshold. The investment threshold is significantly lower when the idiosyncratic volatility is high, which means the optimal investment policy allows the capital stock to fall farther before triggering positive investment. Another way to put it is firms would invest to reach a lower capital stock level when there are investment opportunities. In my baseline calculation, the increase in idiosyncratic productivity shock volatility from $\sigma = 0.1$ to $\sigma = 0.2$ lead to about 40 percent decrease in aggregate investment and about 45 percent fall in the long-run level of aggregate capital stock in steady states. The quantitative implications highlight the importance of volatility in determining capital investment and the accumulation of capital.

Second, the investment threshold function over productivity flattens when the idiosyncratic volatility is higher. This is consistent with the notion that firms are more cautious in undertaking investment projects when the volatility is higher. This finding is similar to

[7]If the capital adjustment features quadratic adjustment cost, the capital stock adjustment process takes time to finish.

results reported by Bloom (2009) and others.

Figure 8 displays the surface of firm value for different levels of productivity and capital stock. The firm's value function is increasing with respect to capital stock size and the level of productivity. These are straightforward implications from the model. The ridge on the surface represents that the value function has kinks when the irreversibility constraint just starts to bind.

5.4 The Transition Dynamics

In this section, I analyze the transition dynamics of the model from the low volatility steady state to the high volatility one. When the idiosyncratic volatility of productivity shocks changes, the transition dynamics are important to answer questions such as: how long does it take for firms to reallocate capital to the new long-run level and how important are these changes to short-run fluctuations?

Figure 9 shows the transition dynamics of consumption, investment, interest rates and the level of capital stock in response to an increase in the idiosyncratic productivity shock volatility from $\sigma = 0.1$ to $\sigma = 0.2$. Investment immediately falls as the idiosyncratic productivity shock volatility goes up. The effect is due to the real option channel. When investment is irreversible, the real option value of waiting increases as the productivity shock volatility shoots up. The fall in investment has a negative effect on the accumulation of capital, thus slowing down the growth of the economy. This results in a lower expected growth path of consumption and lower interest rates as implied by the consumption Euler equation.

An important question to be asked is how does the intertemporal elasticity of substitution $1/\psi$ plays in quantifying the general equilibrium effect. Figure 10 compares the transition path of model under two specification of the ψ. Even though economists haven't reached a concensus about the elasticity of substitution, I consider two benchmark values $\psi = 5$ and $\psi = 0.5$. The first value is widely used by macroeconomists and the latter one is the benchmark setup in the long-run risk literature Bansal and Yaron (2004).

As shown by the graph, the intertemporal elasticity of substitution plays an important role in quantifying the general equilibrium effect. When ψ is larger, there is a stronger response to interest rates, which counteract the partial equilibrium real option effect. The larger is the ψ, the smoother is the transition dynamics of consumption, investment and capital stock.

5.5 Investigating the General Equilibrium Channel

The real interest rates are the medium through which the general equilibrium channel operates. The stronger is the response to interest rates, the larger is general equilibrium effect, which counteracts the partial equilibrium real option effect. Since real interest rates can be empirically measured, the general equilibrium channel may be tested through the response of real interest rates to changes in idiosyncratic volatility.

To explore the aggregate effects of the volatility of firm productivity shock, I consider a baseline regression of real interest rate, idiosyncratic volatility of the form:

$$\Delta r_t = \beta_0 + \beta_1 \Delta \sigma_{\epsilon,t} + \epsilon_{\sigma,t} \tag{34}$$

where r_t is the log of the time t to $t+1$ risk-free rate. The expected real interest rate is given by subtracting the predicted inflation from the log of the nominal interest rate. The nominal interest rate is measured as the annualized Treasury-bill rate from the Federal Reserve Bank of St.Louis (TB3MS) series. Annual inflation is calculated as the log of Consumer Price Index (CPI) in December in year t, divided by CPI in December of year $t-1$. This is modeled as using an ARMA(1,1) process, and the predicted value is used as the estimate of expected inflation as in Constantinides and Ghosh (2011).

I take the first difference of interest rates and the level of idiosyncratic volatility because they seem to have non-stationary components. It is found that there is a modest negative relationship between the volatility of firm level productivity shock and the real risk-free rate. The regression coefficient is -0.16 with a t-statistics of -1.6. Table 3 presents the summary statistics and Figure 1 and 11 plots the time-series of idiosyncratic volatility, real and nominal interest rates. I also consider using longer term interest rates, such as 10-year Treasury rate, as the proxy for interest rates. The quantitative results remain mostly similar. This is a moderate degree of negative relationship between changes in interest rates and idiosyncratic volatility.

The next exercise I consider is to compute the model implied impulse responses of interest rates to changes in idiosyncratic volatility. I calculate the elasticity of intertemporal substitution to match the model implied impulse response to the empirical response of interest rates. That inverse intertemporal elasticity of substitution is computed is to be 1.4, which suggests that the general equilibrium effect is modest but no big enough to largely offset the partial equilibrium effect. In terms of the response of consumption, investment, the general equilibrium effect tamer the partial equilibrium effect by about 20%.

6 Conclusion

This paper documents robust evidence on the upward trend in idiosyncratic volatility of productivity shocks. While the volatility of firm cash flow growth, firm stock return are documented to display such upward trend, the increase in the volatility of productivity shocks may be the fundamental reason behind the rise in firm-level risk. This finding contributes to the literature on the firm-level risk.

To quantitatively investigate the consequences of the upward trend in idiosyncratic productivity shock volatility, I build a dynamic general equilibrium model with firm heterogeneity. The increase in idiosyncratic volatility has a significant negative effect on the firm investment and capital accumulation both at the short-run and long-run. The doubling of idiosyncratic productivity shocks volatility could lead to about 40% decrease in long-run aggregate investment and capital stock.

The short-run effects on investment and capital expenditure work through two channels. The first effect works through is the partial equilibrium real option effect. When the volatility of productivity shocks is high, the real option value of waiting increases and firms thus delay their investments. The second channel works through the general equilibrium effect of interest rates on investment. In equilibrium, the fall in aggregate investment corresponds to expected future decline in consumption growth and thus lower real interest rates. The decrease in interest rates would spur investment and thus counteract the partial equilibrium real option effect. Under different parameters for the intertemproal elasticity of substitution, the size of general equilibrium effect varies. The smaller is the intertemproal elasticity of substitution, the larger is the counteracting general equilibrium effect, the slower is the speed of transition to new steady states. Pinning down the magnitude of the general equilibrium effect remains an important open question and deserves further research.

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Tables and Figures

Table 1: The Dynamics of Idiosyncratic Volatility

Variables	σ_{ϵ_t}	Variables	$\Delta\sigma_{\epsilon_t}$
$\sigma_{\epsilon_{t-1}}$	0.91	$\Delta\sigma_{\epsilon_{t-1}}$	-0.002
	(0.05)		(0.15)

The table reports AR(1) coefficients for the level the change of idiosyncratic volatility. Standard errors are reported in parentheses.

Table 2: Calibration

Parameter	Description	Value
Panel A: Preferences		
ρ	Time Preference Rate	0.05
φ	Inverse of Intertemporal Elasticity of Substitution	0.3
Panel B: Technology		
α	Capital share	0.33
δ	Depreciation rate of capital stock	0.10

This table reports the parameter values used in the quarterly calibration of the model. The table is divided into two categories: preferences and technology.

Table 3: The Impact of Idiosyncratic Volatility on the Risk-free Interest Rate

	1963-2015
Variables	$\Delta\sigma_{\epsilon_t}$
Δr_{3m}	-0.16
	(0.10)
Observations	51
t-statistic	-1.6

This table reports the coefficient of regressing the change of real interest rates on the change of idiosyncratic volatility. The sample period is is from 1963 to 2015.

Figure 1: The Idiosyncratic Volatility of Productivity Shocks

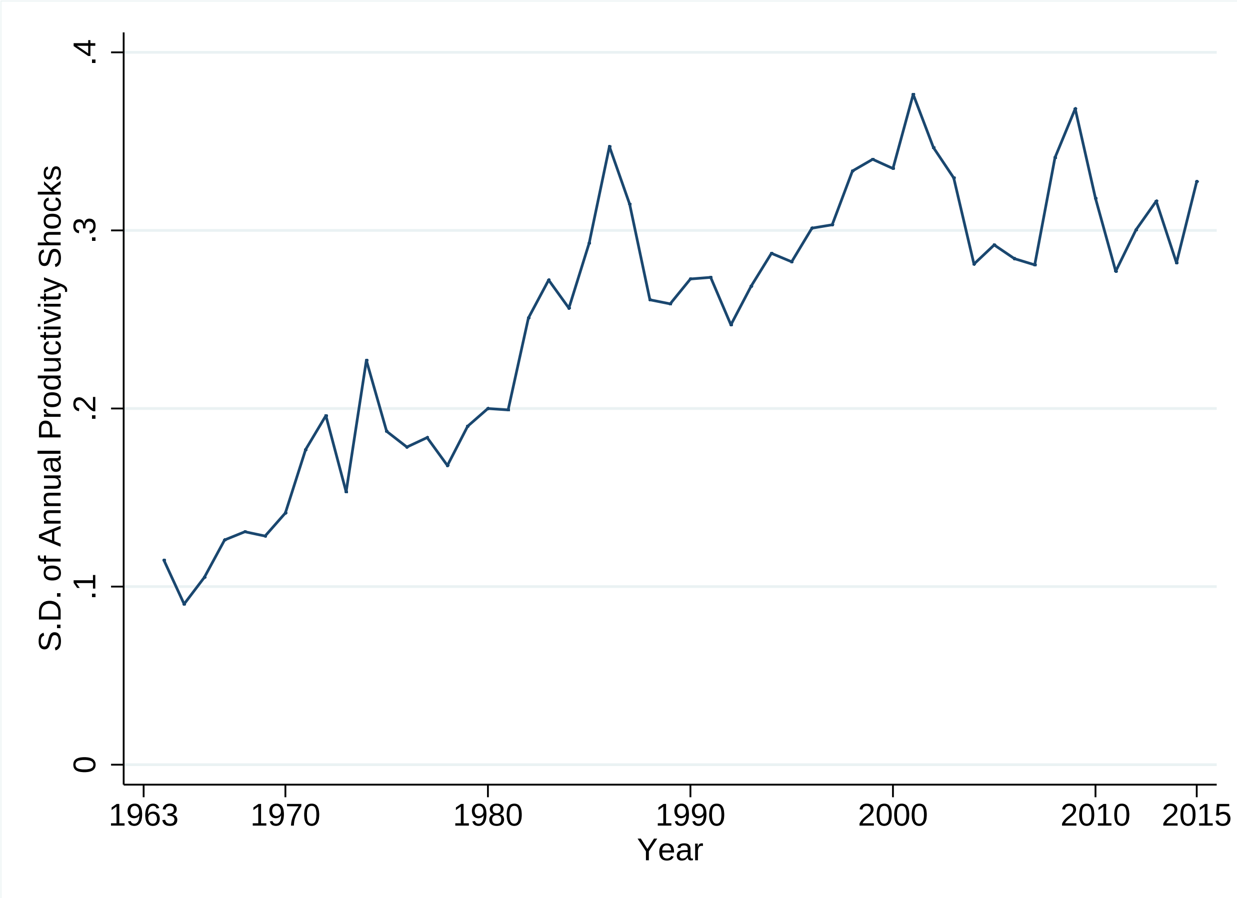


Figure 2: The Idiosyncratic Volatility of Productivity Shocks

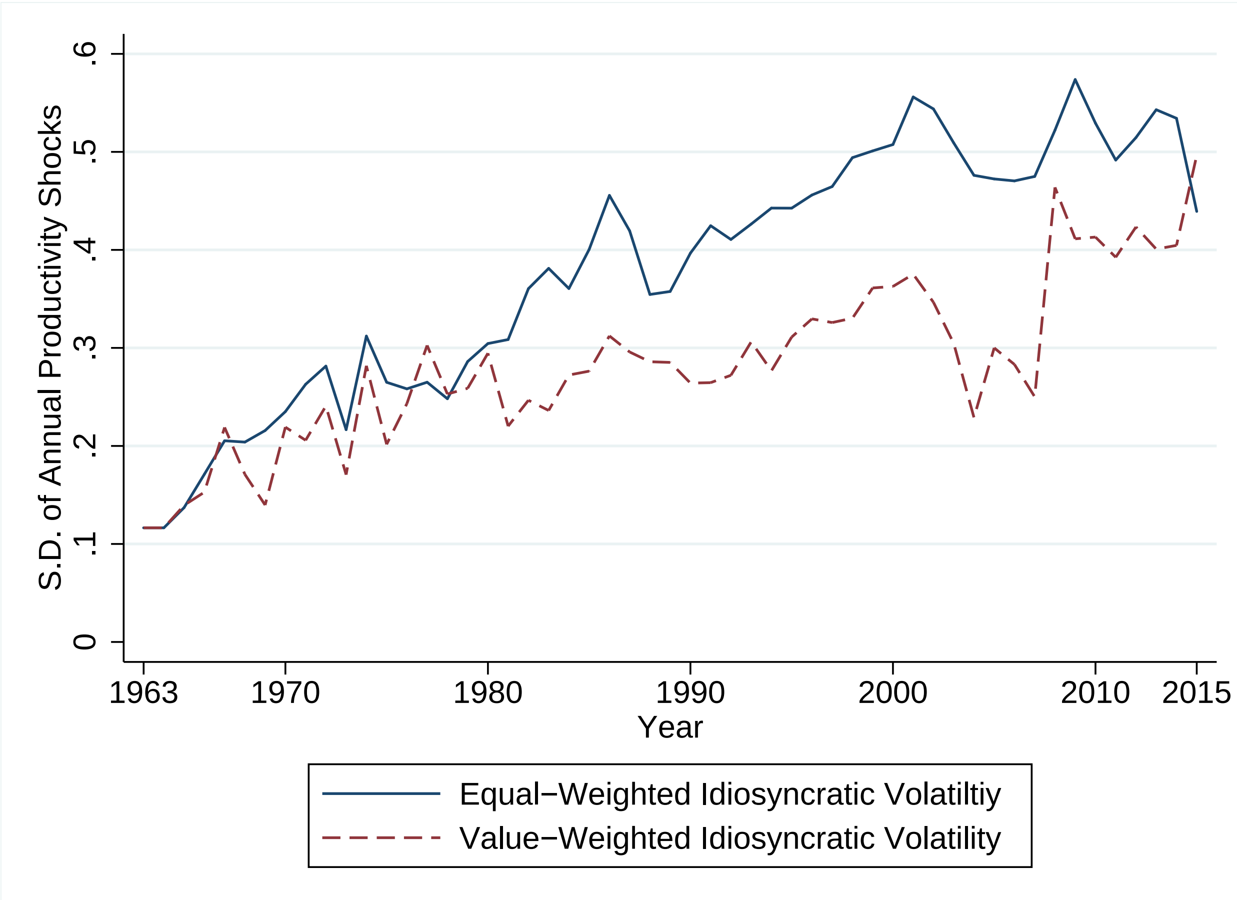
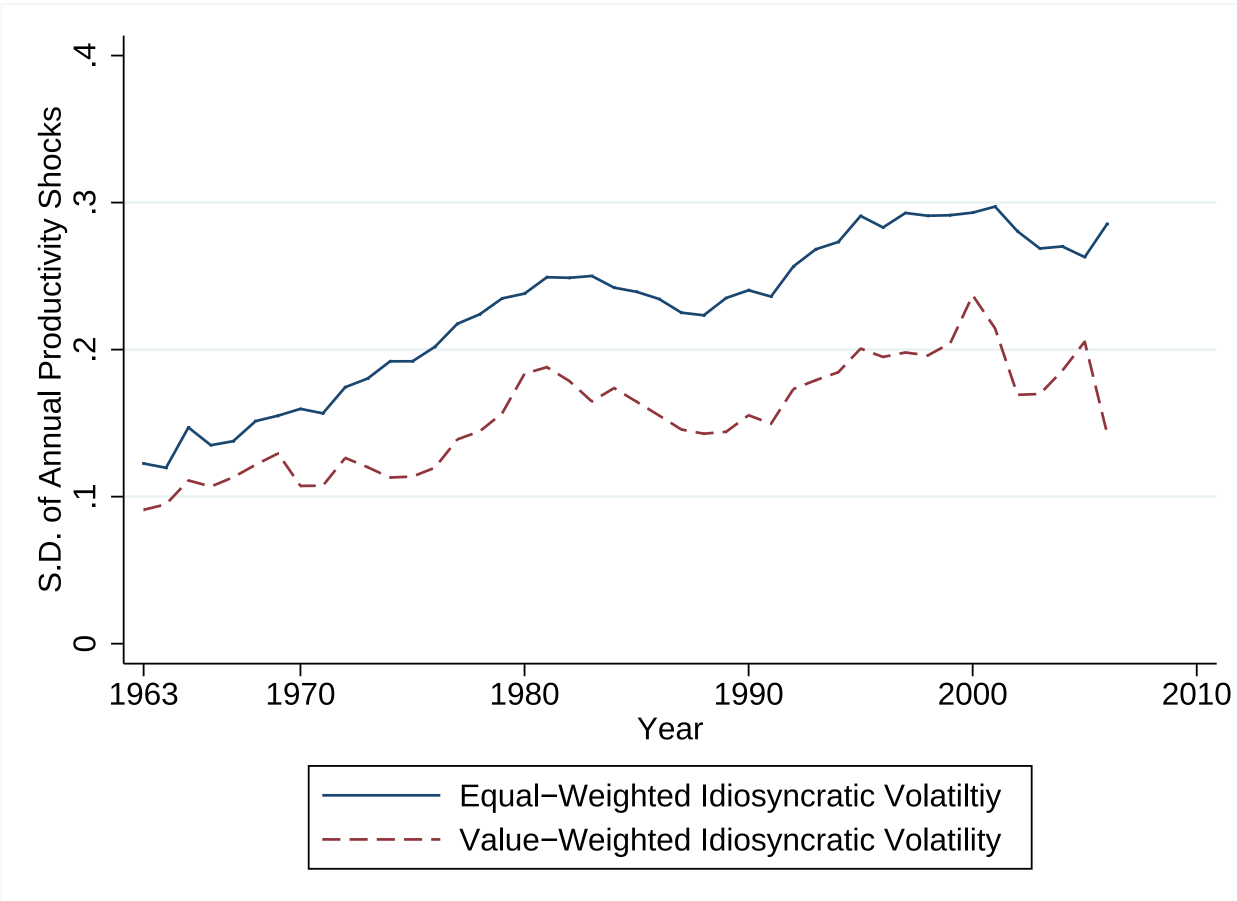
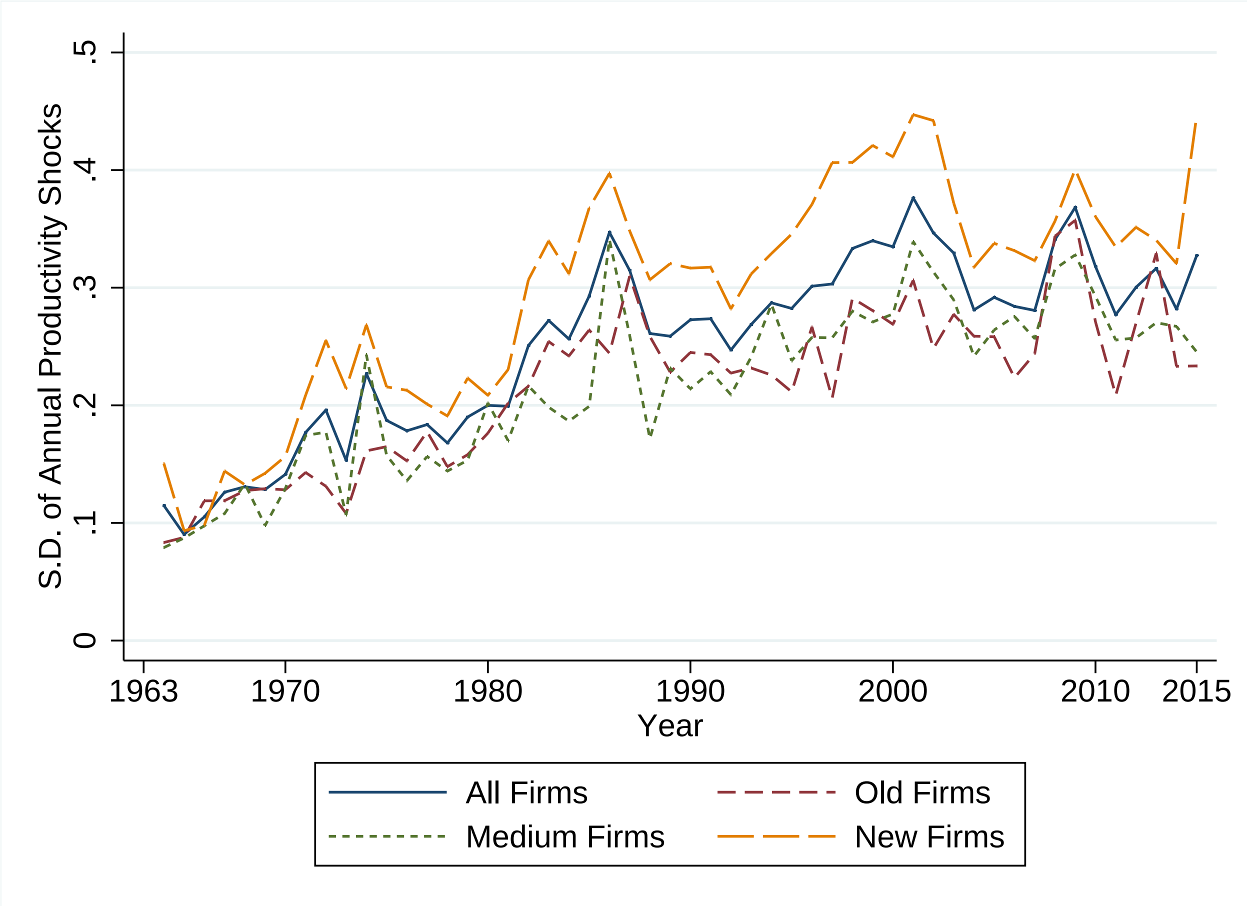


Figure 3: The Idiosyncratic Volatility of Productivity Shocks Using Rolling S.D.



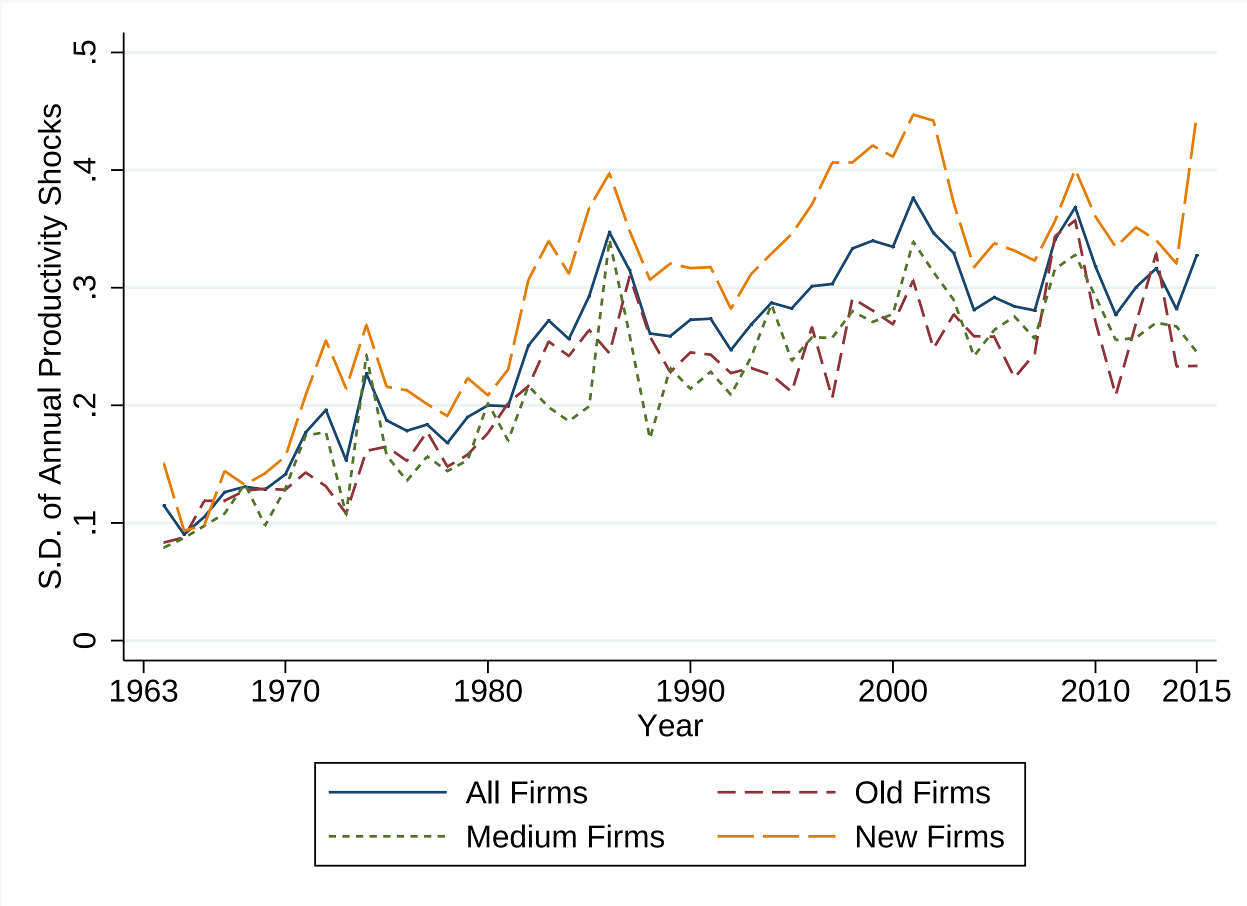
This figures the time series for the idiosyncratic productivity shocks volatility using rolling-window standard deviations.

Figure 4: The Idiosyncratic Volatility of Productivity Shocks Controlling for Size



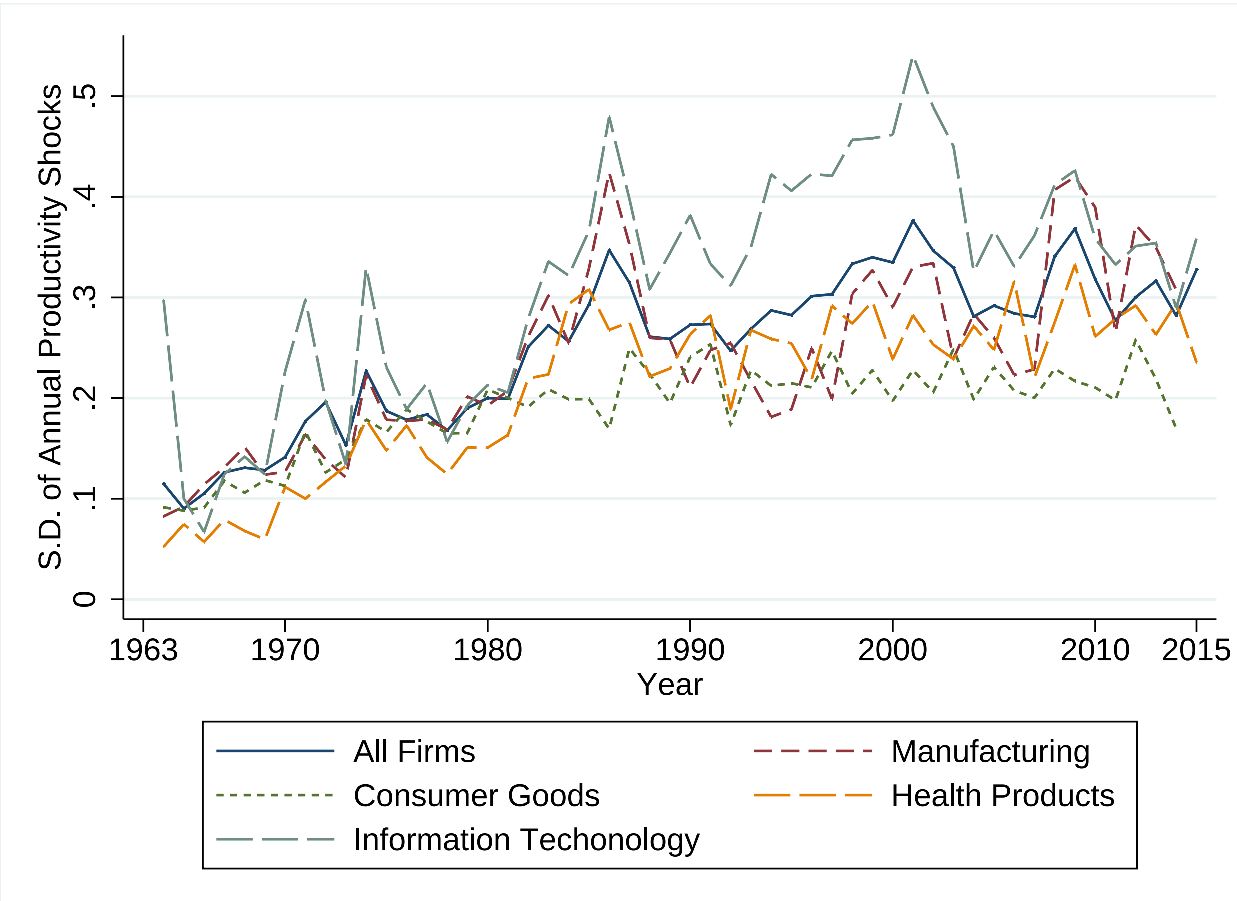
This figures the time series for the idiosyncratic productivity shocks volatility for firms in different size groups.

Figure 5: The Idiosyncratic Volatility of Productivity Shocks Controlling for Age



This figures the time series for the idiosyncratic productivity shocks volatility for different age groups.

Figure 6: The Idiosyncratic Volatility of Productivity Shocks Controlling for Sectors



This figures the time series for the idiosyncratic productivity shocks volatility for different sectors.

Figure 7: The Investment Threshold

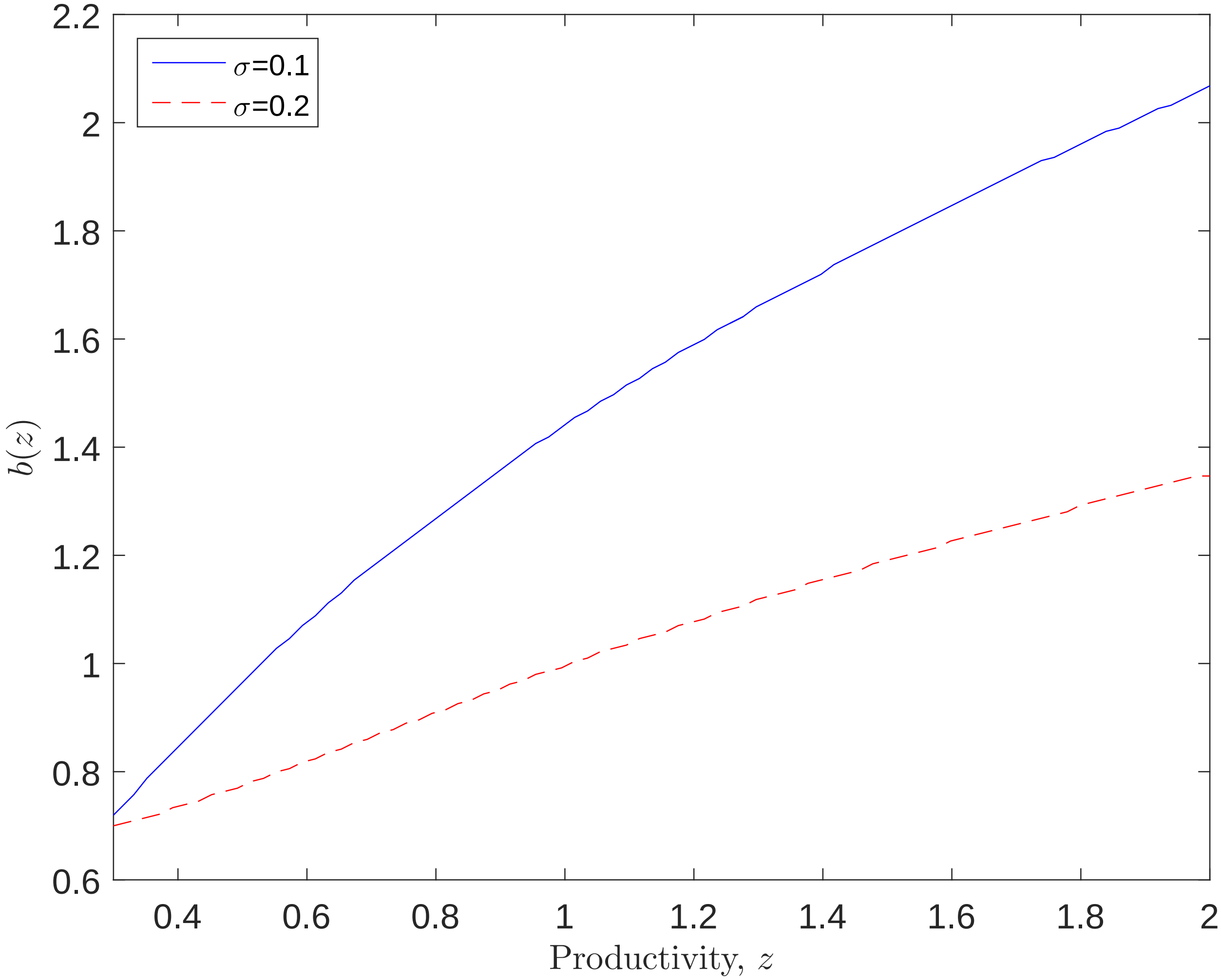


Figure 8: The Value of Firms over Productivity and Capital

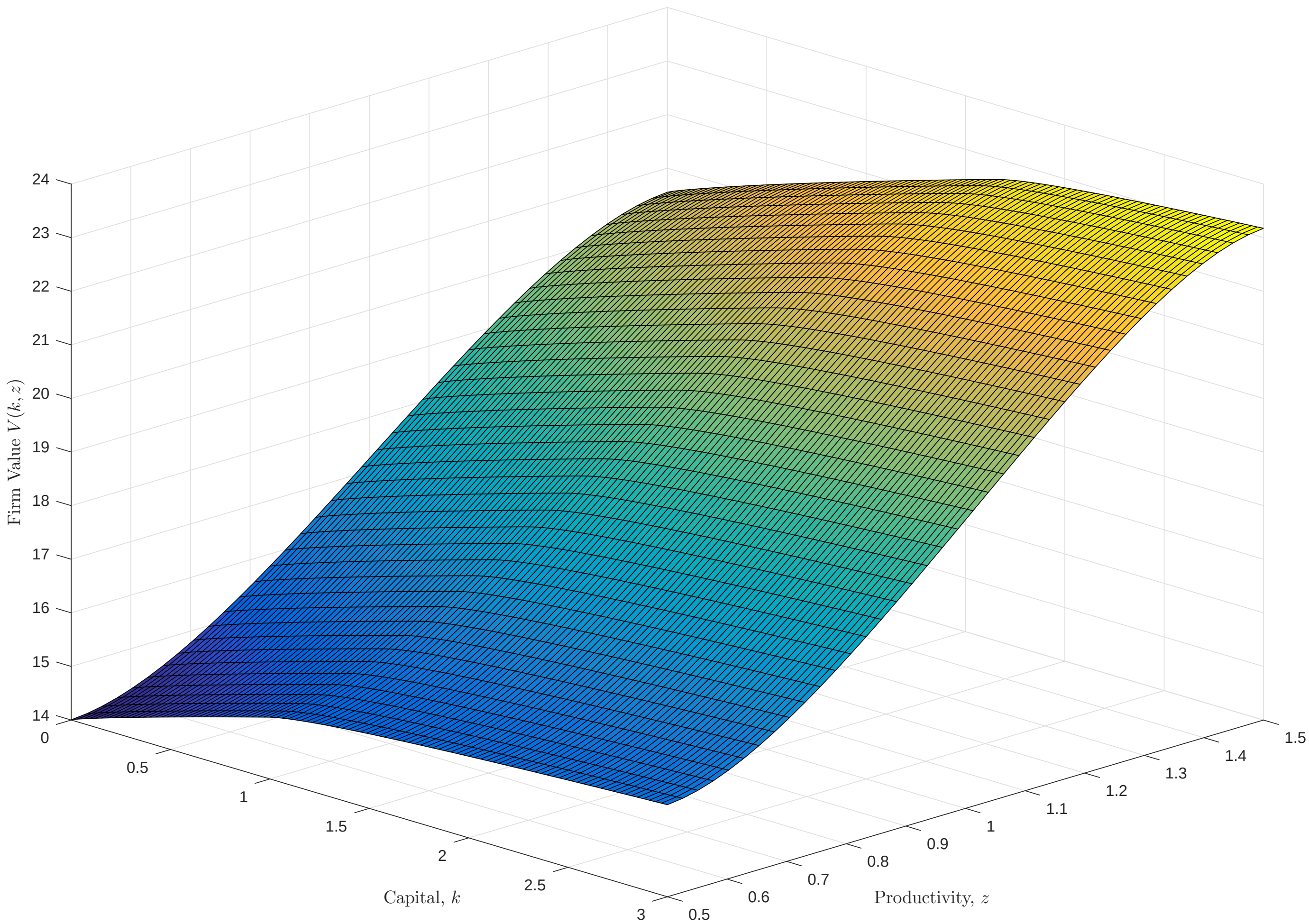
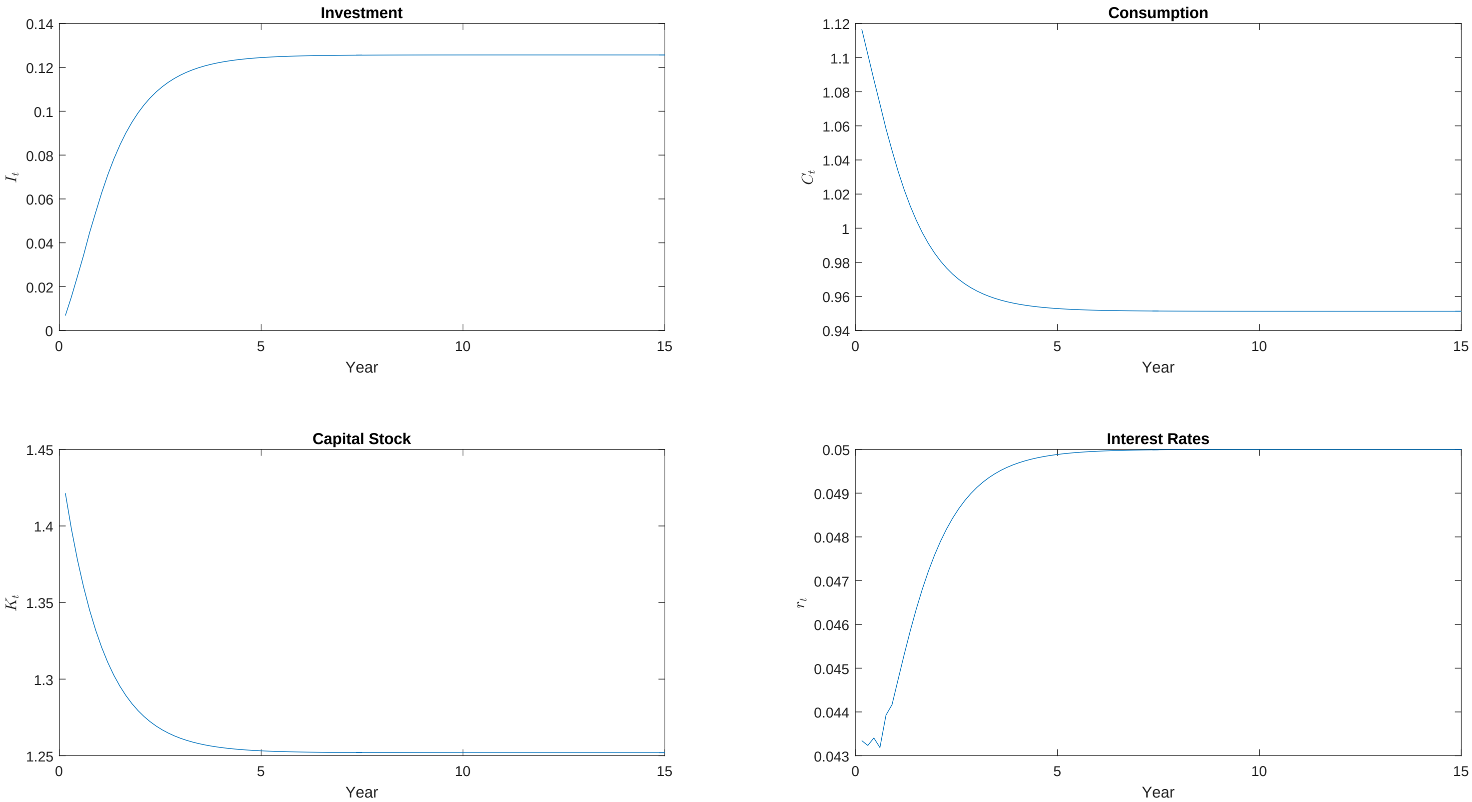
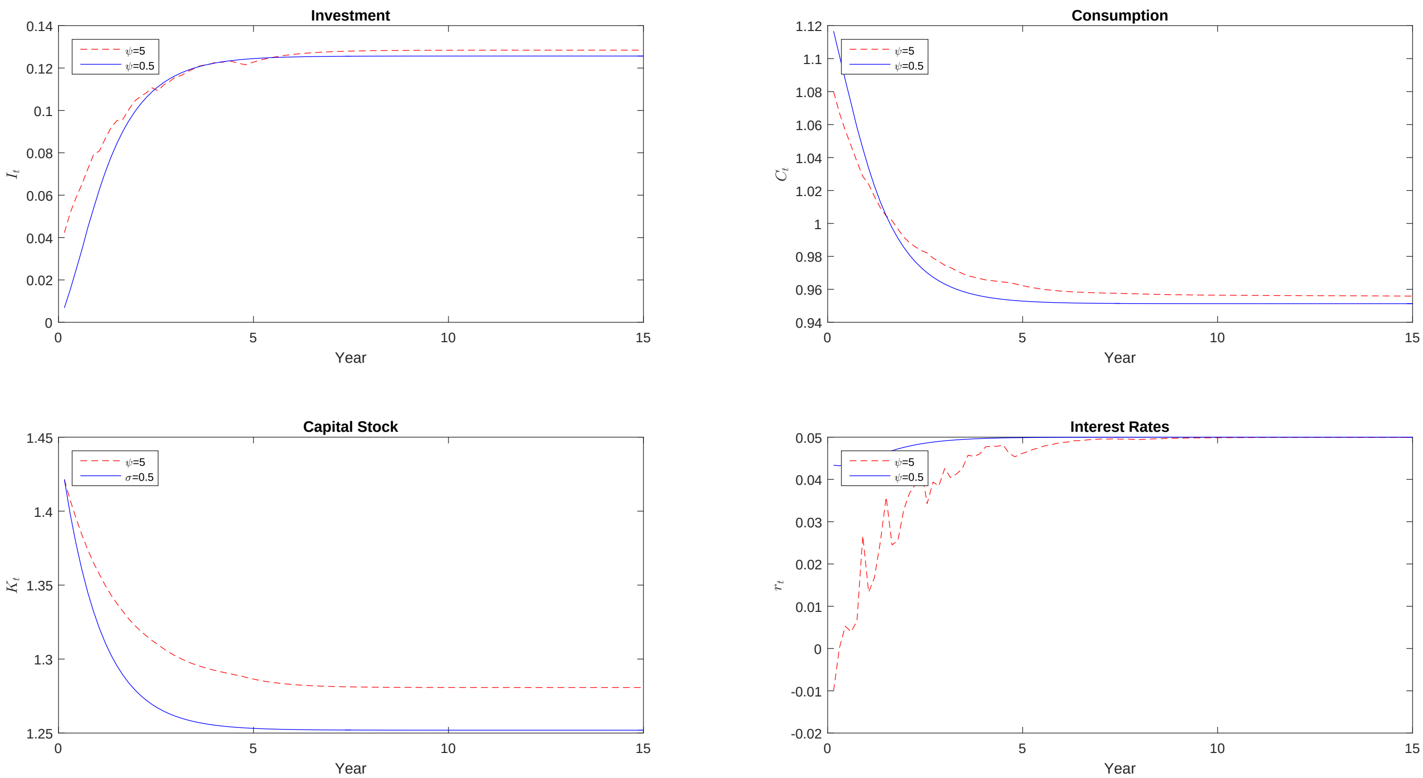


Figure 9: The Transition Dynamics of Consumption, Investment, Capital Stock Level and Interest Rates



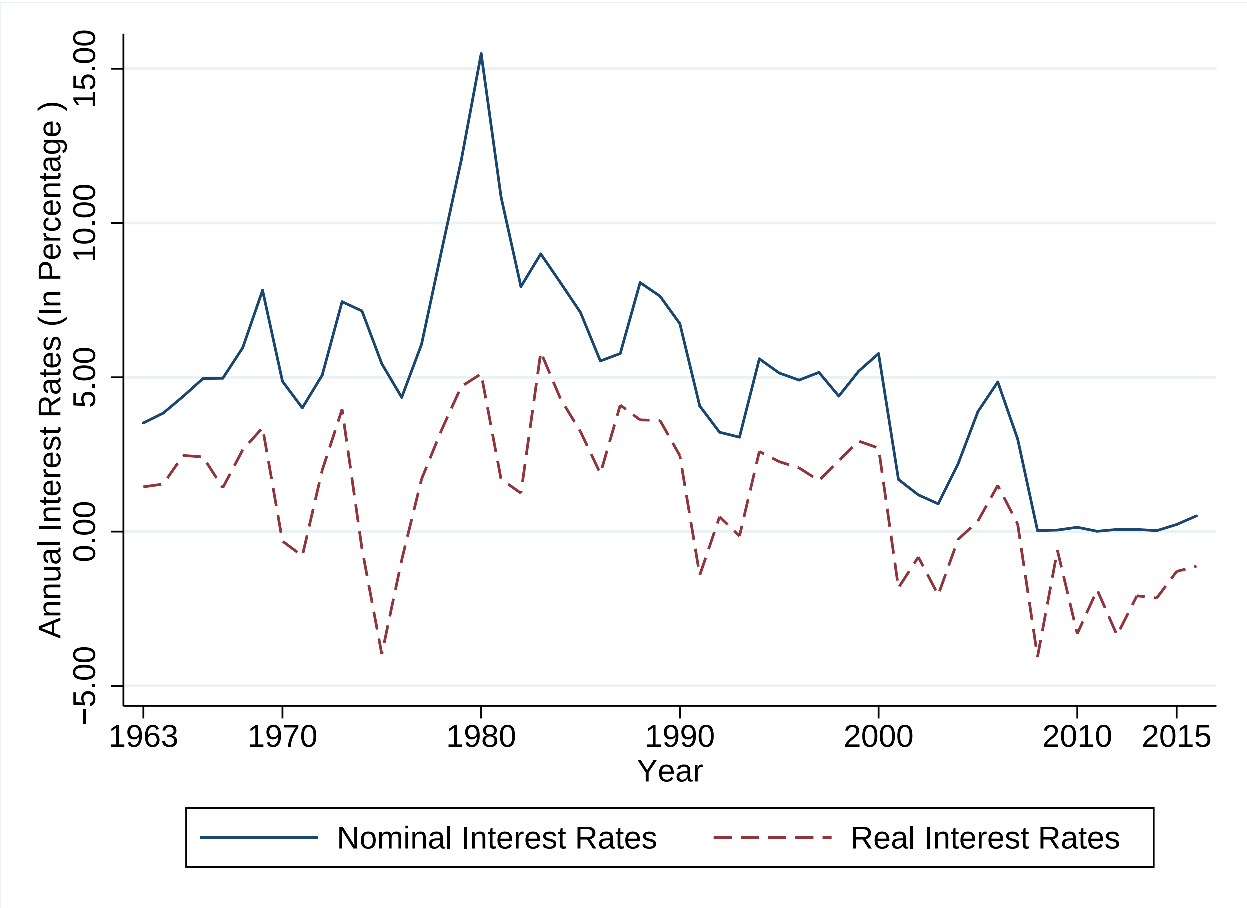
The figures plots the transition dynamics of investment, consumption, capital stock and interest rates to a unexpected permanent change of $\sigma = 0.1$ to $\sigma = 0.2$. The units on the horizon axis represent the time after which the change happens measured in years. The inverse of the intertemporal elasticity of substitution ψ equals to 0.5.

Figure 10: The Transition Dynamics and Elasticity of Intertemporal Substitution



The figures plots the transition dynamics of investment, consumption, capital stock and interest rates to a unexpected permanent change of $\sigma = 0.1$ to $\sigma = 0.2$. The units on the horizon axis represent the time after which the change happens measured in years. The inverse of the intertemporal elasticity of substitution ψ equals to 0.5 and 5.

Figure 11: The Time Series of Nominal and Real Interest Rates



This figures plot the time-series for the nominal and real interest rates, which are measured in percentage points.

7 Appendix

Empirical Part

In this section, I present the empirical evidence about the volatility of firm-level productivity.

Firm Level Data

The data source I use to estimate firm level productivity measure is Compustat. I use the Compustat fundamental annual data from 1962 to 2009. As it is common in the literature (Belo et al. (2014), Imrohoroglu and Tüzel (2014)), I delete observations of financial firms (SIC classification between 6000 and 6999) and regulated firms (SIC classification between 4900 and 4999). My sample for production function estimation is comprised of all remaining firms in Compustat that have positive data on sales, total assets, number of employees, gross property, plant, and equipment, depreciation, accumulated depreciation, and capital expenditures. The sample is an unbalanced panel with approximately 12,750 distinct firms spanning the years between 1962 and 2009. Following Fama and French (1992), we start our sample in 1962 since Compustat data for earlier years have a serious selection bias.

The key variables for estimating firm level productivity in our benchmark case are the firm level value added, employment, and physical capital. Firm level data is supplemented with price index for Gross Domestic Product as deflator for the value-added and price index for private fixed investment as deflator for investment and capital, both from Bureau of Economic Analysis, and national average wage index from the Social Security Administration.

Value added (y_{it}) is computed as Sales - Materials, deflated by the GDP price deflator. Sales is net sales from Compustat (SALE). Materials is measured as Total expenses minus Labor expenses. Total expenses is approximated as [Sales-Operating Income Before Depreciation and Amortization (Compustat (OIBDP))]. Labor expenses is calculated by multiplying the number of employees from the Social Security Administration. The stock of labor (l_{it}) Compustat (EMP) by average wages from the Social Security Administration. The stock of labor l_{it} is measured by the number of employees from Compustat (EMP). These steps lead to our value added definition that is proxied by Operating Income before Depreciation and Amortization + labor expenses.

Capital stock (k_{it}) is given by gross property, plant, and equipment (PPEGT) from Compustat, deflated by the price deflator for investment following the methods of Hall (1990) and Brynjolfsson and Hitt (2003). Since investment is made at various times in the past, we need to calculate the average age of capital at every year for each company and apply the appropriate deflator (assuming that investment is made all at once in year [Current Year- Age]). Average age of capital stock is calculated by dividing accumulated depreciation (Gross PPE - Net PPE, from Compustat (DPACT)) by current depreciation. The resulting capital stock is lagged by one period to measure the available capital stock at the beginning of the period.

7.1 Firm Level Productivity

In this paper, the production function to be estimated is given by

$$y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \eta_{it} \tag{35}$$

where y_{it} is the log of value added for firm i in period t; l_{it} and k_{it} are log values of labor and capital of the firm, respectively; ω_{it} is the productivity; and η_{it} is an error term not known by the firm or the econometrician. I consider the semi-parametric procedure suggested by Olley and Pakes (1996) to estimate the parameters of this production function. This method has been recently used by Imrohoroglu and Tüzel (2014) to estimate firm level productivity. The major advantage of this approach over more traditional estimation techniques such as the ordinary least squares is its ability to control for selection and simultaneity biases and deal with the within firm serial correlation in productivity that troubles many production function estimates.

Olley and Pakes (1996) assumes that productivity w_{it}, is observed by the firm before the firm makes some of its factor input decisions, which give rise to the simultaneity problem. Labor, l_{it} is the only variable input, i.e, its value can be affected by the current productivity, ω_{it}. The other input, k_{it}, is a fixed input at time t, and its value is only affected by the conditional distribution of w_{it} at time $t-1$. Consequently, w_{it} is a state variable that affects firms' decision making where firms that observe a positive probability shock in period t will invest more in capital, i_{it}, and hire more labor, l_{it}, in that period. The solution to the firm's optimization problem results in the equation for i_{it}:

$$i_{it} = i(\omega_{it}, k_{it}) \tag{36}$$

where both i and j are strictly increasing in ω. The inversion of the equations yield:

$$\omega_{it} = h(i_{it}, k_{it}) \tag{37}$$

where h is strictly increasing in i_{it}. We can define $\phi_{it} = \beta_0 + \beta_k k_{it} + h(i_{it}, k_{it})$. Substituting ϕ_{it} into (35) yields

$$y_{it} = \beta_l l_{it} + \phi_{it} + \eta_{it} \tag{38}$$

where we approximate ϕ_{it} with a second order polynomial series in capital and investment. This first stage estimation results in an estimate for $\hat{\beta}_l$ that controls for the simultaneity problem. In the second stage, consider the expectation of $y_{i,t+1} - \hat{\beta}_l l_{i,t+1}$ on information at time t and survival of the firm:

$$\begin{aligned} \mathbb{E}_t(y_{i,t+1} - \hat{\beta}_l l_{i,t+1}) &= \beta_0 + \beta_k k_{i,t+1} + \mathbb{E}_t(\omega_{it+1} | \omega_{it}, survival) && (39) \\ &= \beta_0 + \beta_k k_{i,t+1} + g(\omega_{it}, \hat{P}_{\text{survival},t}) && (40) \end{aligned}$$

The survival probability is estimated via a probit of a survival indicator variable on a polynomial expression containing capital and investment. We fit the following equation by nonlinear least squares:

$$y_{i,t+1} - \hat{\beta}_l l_{i,t+1} = \beta_k k_{i,t+1} + \rho\omega_{it} + \tau \hat{P}_{\text{survival},t} + \eta_{i,t+1} \tag{41}$$

where ω_{it} is given by $w_{it} = \phi_{it} - \beta_0 - \beta_k k_{it}$ and is assumed to follow an AR(1) process. At the end of this stage, $\hat{\beta}_l$ and $\hat{\beta}_k$ are estimated. Finally, productivity is measured by:

$$p_{it} = \exp(y_{it} - \beta_0 - \hat{\beta}_l l_{it} - \hat{\beta}_k k_{it}) \tag{42}$$

The estimates for the production function are summarized in the Table A.1. The production function estimated display nearly constant returns to scale and are insensitive to the samples used. The properties of firm level productivities are summarized in Table A.2

Table A.1: Estimates for Production Function Parameters

Sample	Obs	β_k	β_l	$\beta_k + \beta_l$
Compustat	103707	0.228	0.746	0.9740
CRSP/Compustat	84130	0.2614	0.7319	0.9933

Notes: I use both the Compustat and the Compustat/CRSP merged dataset. The production function estimates displayed are stable and insensitive to the sample chosen. The production function displays constant returns to scale.

Table A.2: Summary Statistics for Firm Level Productivity

Sample	Obs	Mean	Std.Dev.	Skewness	Kurtosis	Percentiles		
						0.05	0.50	0.95
Compustat	103707	0.068	0.043	37.70	3235.597	0.036	0.063	0.113
CRSP/Compustat	84130	0.072	0.049	42.82	3625.37	0.039	0.066	0.120

8 Solution of the Model When Productivity follows Geometric Brownian Motion

This section develops an analytical solution of the model when productivity follows a Geometric Brownian motion. The analysis draws on Bertola and Caballero (1994) and

Stokey (2008).

8.1 Firm Investment Problem

This section provides details to characterize the firm investment rule in steady state. First, I reduce the dimensionality of the optimization problem by the homogeneity property of value function. By the virtue of this simplification, I could obtain closed form solution for firm investment problem when the idiosyncratic productivity follows. The aggregated investment function can then be computed.

8.1.1 Exploiting Homogeneity

In my model setup, the profit function is homogeneous of degree one,

$$\Pi(K,Z) = Z\pi(K/Z)$$

where $\pi(\hat{K}) \equiv \Pi(\hat{K},1)$. Also define the piecewise linear function

$$\rho(I) = \begin{cases} I, \ I \geq 0 \\ 0, \ I < 0 \end{cases}$$

The firm's problem can then be written as

$$V(K_t, Z_t) = \max_{\{I(t)\}} \mathbb{E}_t \left[\int_0^\infty e^{-ru} \{ [Z\pi(K/Z) - K[\rho(I/K)] \} du] \right] \tag{43}$$

subject to

$$\begin{aligned} \frac{dK}{K} &= (\frac{I}{K} - \delta)dt \\ dZ/Z &= \mu dt + \sigma dW \end{aligned} \tag{44}$$

The associated HJB equation is

$$\begin{aligned}rV =&Z\pi(K/Z) - \delta K V_K + \mu V_Z + \frac{1}{2}\sigma^2 V_{ZZ}\\&+ \max_I \{V_K I - K\rho(I/K)\}\end{aligned} \tag{45}$$

where V and its derivatives are evaluated at $(K/Z, 1)$. This second order PDE can be written as an ODE by exploiting homogeneity. The return function and constraints in (43) and (44) are homogeneous of degree one in (K, Z, I). Hence V is homogeneous of degree one in (K, Z), and the optimal policy is homogeneous in the sense that if the stochastic process I is optimal for the initial conditions (K_t, Z_t), then for any $\lambda > 0$ the process λI^* is optimal for the initial conditions $(\lambda K_t, \lambda Z_t)$. Define the ratios $\mathcal{K} \equiv K/Z$ and $\mathcal{I} \equiv I/K$, and the intensive form of the value function $v(\mathcal{K}) \equiv V(\mathcal{K}, 1)$, $\mathcal{K} \geq 0$. Then

$$V(K, Z) = Zv(K/Z), \quad \text{all } K, Z$$

Thus,

$$V_K = v', \quad V_Z = v - \mathcal{K}v', \quad V_{ZZ} = \mathcal{K}^2 \frac{1}{Z} v''$$

Substituting for V and its derivatives in (45) gives the HJB equation in the intensive form

$$(r - \mu)v = \pi(\mathcal{K}) - (\delta + \mu)\mathcal{K}v' + \frac{1}{2}\sigma^2 \mathcal{K}^2 v'' + \mathcal{K} \max_{\mathcal{I} \geq 0} [v'\mathcal{I} - \rho(\mathcal{I})] \tag{46}$$

The coefficient on v in the normalized HJB equation is $r - \mu(Z)$. Since the investment problem is a special case in which the investment is defined by a threshold, we could obtain the result below. If $K < b(Z)$, the firm makes a discrete investment of size $b(Z) - K$, so below the threshold the value function is

$$V(K, Z) = V[b(Z), Z] + b(Z) - K, \quad K < b(Z)$$

Therefore, investment is just sufficient to keep K from falling below $b(Z)$. The region above $b(Z)$ is the inaction region. In this region the value function satisfies the HJB equation in the intensive form

$$(r-\mu)v = \pi(\mathcal{K}) - (\delta+\mu)\mathcal{K}v' + \frac{1}{2}\sigma^2\mathcal{K}^2 v'' \tag{47}$$

In the region where the firm makes discrete investments

$$v(\mathcal{K}) = v(b^*) + (b^* - \mathcal{K}), \quad \mathcal{K} < b^* \tag{48}$$

The optimal threshold has the form $b(Z) = b^* Z$ where the constant b^* must be determined. Thus, by exploiting the homogeneity property of the value function, I reduce the second order partial differential equation (PDE) of (10) into a normalized HJB equation in the form of an ordinary differential equation (ODE).

9 Closed Form Solutions under Geometric Brownian Motion Productivity Process

9.1 Solving the ODE

When $\Pi(K,Z) = K^\alpha Z^{1-\alpha}$ and $dZ/Z = \mu dt + \sigma dW_t$, the intensive form of the HJB equation is

$$(r-\mu)v = k^\alpha - (\delta+\mu)kv' + \frac{1}{2}\sigma^2 k^2 v'', \quad k > b^* \tag{49}$$

The normalized HJB equation (49) is a standard second order linear nonhomogeneous differential equation. All solutions have the form

$$v(\mathcal{K}) = v_p(\mathcal{K}) + \alpha_1 h_1(\mathcal{K}) + \alpha_2 h_2(\mathcal{K}), \quad \mathcal{K} > b^*$$

where $v_p(\mathcal{K})$ is any particular solution, $h_i(\mathcal{K})$, $i = 1, 2$, are homogeneous solutions. It is easy to verify that a particular solution has the form of

$$v_p(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^\alpha$$

.

The homogeneous solutions are $h_i(\mathcal{K}) = \mathcal{K}^{R_i}$, $i = 1, 2$, where R_1 and R_2 are the roots of the quadratic

$$0 = (r - \mu) + (\delta + \mu)R - \frac{1}{2}\sigma^2 R(R - 1)$$

The assumption $r > \mu$ insures the roots are real and of opposite sign. Label them $R_1 < 0 < R_2$. Therefore, all solutions can be written as

$$v(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^{\alpha} + \alpha_1 \mathcal{K}^{R_1} + \alpha_2 \mathcal{K}^{R_2}, \quad k > b^*$$

where the constants α_1 and α_2 must be determined. Since there is no upper threshold,

$$\lim_{\mathcal{K} \to \infty} \left(v(\mathcal{K}) - \frac{1}{\eta}\mathcal{K}^{\alpha} = 0 \right)$$

reflecting the fact that as $\mathcal{K} \to \infty$, the time until investment is positive becomes arbitrarily long, with probability arbitrarily close to one. Since $R_1 < 0 < R_2$, this condition holds if and only if $\alpha_2 = 0$. Let R (without a subscript) denote the negative root, so the value function has the form

$$v(\mathcal{K}) = \begin{cases} \mathcal{K}^{\alpha}/\eta + a_1 \mathcal{K}^{R}, & \mathcal{K} \geq b^* \\ v(b^*) - (b^* - \mathcal{K}) & 0 \leq \mathcal{K} < b^* \end{cases}$$

where R satisfies

$$\begin{aligned} R &\equiv \frac{1}{\sigma^2}(m - D) \\ D &\equiv \left[m^2 + 2\sigma^2(r - \mu) \right]^{1/2} \\ m &\equiv \delta + \mu + \frac{1}{2}\sigma^2 \end{aligned}$$

It remains to determine a_1 and b^*. The smooth pasting condition, $\lim_{k \downarrow b^*} v'(\mathcal{K}) = P$ suggests that

$$a_1 = \frac{1}{R}\left[P(b^*)^{1-R} - \frac{\alpha}{\eta}(b^*)^{\alpha-R}\right]$$

The super contact condition $\lim_{k\downarrow b^*} v''(\mathcal{K}) = 0$ requires that

$$b^* = A^{1/(\alpha-1)}$$

where

$$A \equiv \frac{\eta}{\alpha}\frac{1-R}{\alpha-R}$$

Write A, η, m, D, and R as functions of σ^2 and use to find that

$$\frac{\partial A/\partial\sigma^2}{A} = \frac{\partial\eta/\partial\sigma^2}{\eta} + \frac{(1-\alpha)\partial R/\partial\sigma^2}{(1-R)(\alpha-R)}$$

Clearly $\partial\eta/\partial\sigma^2 > 0$, so the first term is positive. The second term is also positive if $\partial R/\partial\sigma^2 > 0$ It can be shown that

$$\frac{\partial R}{\partial\sigma^2} = \frac{D-m}{2D\sigma^2}\left(\frac{D-m}{\sigma^2}+1\right) > 0$$

Thus when investment is irreversible a higher variance σ^2 leads to a lower investment threshold b^*. That is the optimal policy allows the ratio of the capital stock to demand to fall farther before triggering positive investment. In irreversible case greater uncertainty reduces investment.

Proposition 3. *In the steady state of the model, the value function associated with* (8) *has the form*

$$\begin{aligned} V(K,Z) &= Zv(\mathcal{K}) \\ v(\mathcal{K}) &= v(b^*) + (b^* - \mathcal{K}), \quad \mathcal{K} < b^* \\ v(\mathcal{K}) &= v_p(\mathcal{K}) + \alpha_1 h_1(\mathcal{K}), \quad \mathcal{K} \geq b^* \end{aligned}$$

where $\mathcal{K} \equiv K/Z$, $v(\mathcal{K})) \equiv V(\mathcal{K},1)$, $v_p(\mathcal{K}) = \frac{1}{\eta}\mathcal{K}^\alpha$ *and* $h_1(\mathcal{K}) = \mathcal{K}^{R_1}$. *The constants are* *defined as*

$$\begin{aligned}
\eta &\equiv (r-\mu) + \alpha(\delta+\mu) - \alpha(\alpha-1)\frac{1}{2}\sigma^2 \\
a_1 &\equiv \frac{1}{R}\left[P(b^*)^{1-R} - \frac{\alpha}{\eta}(b^*)^{\alpha-R}\right] \\
A &\equiv \frac{\eta}{\alpha}\frac{1-R}{\alpha-R} \\
b^* &\equiv (AP)^{1/(\alpha-1)} \\
R &\equiv \frac{m-D}{\sigma^2} \\
D &\equiv [m^2 + 2\sigma^2(r-\mu)]^{1/2} \\
m &\equiv \sigma + \mu + \frac{1}{2}\sigma^2
\end{aligned}$$

The investment threshold has the form $b(Z) = b^*Z$. The investment function therefore follows

$$dL_t = \begin{cases} 0 & \text{if } K > b(Z) \\ b(Z) - K = b^*Z - K & \text{if } K \leq b(Z) \end{cases}$$

The main insight from the above formula is when investment is irreversible a higher variance σ^2 leads to a lower investment threshold b^*. That is because the optimal policy allows the ratio of the capital stock to demand to fall farther before triggering positive investment. In irreversible case, greater uncertainty reduces investment.

9.2 Cross-sectional Distribution of Firm Capital Growth Rates

To study the aggregate investment, it is necessary to track the whole cross-sectional density of firm capital growth. It is useful to introduce a few notations to characterize the cross-sectional distribution of investments. Let $k_{i,t} \equiv \log(K_{i,t})$ denote the log capital for firm i. I use $k_t^* = \log b^* + \log z$ to denote the log of "desired" capital if were no irreversibility constraint at time t. Also define $s_t \equiv k_t - k_t^*$ as the difference fo the log capital stock from the log “desired” capital stock. By Ito’s lemma

$$dk_t^* = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

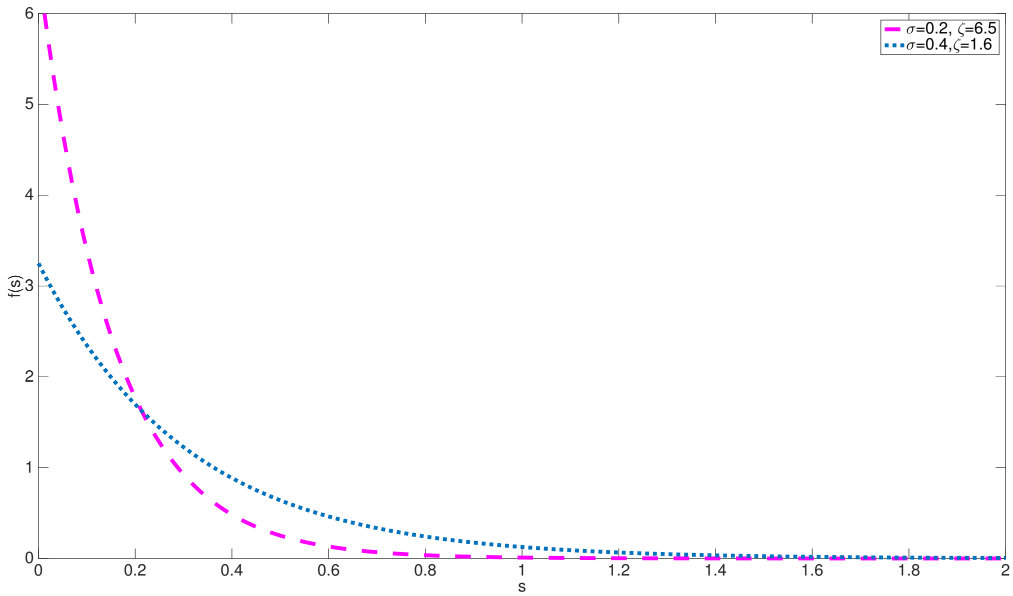
$$ds_t = dk_t - dk_t^* = \begin{cases} 0 & \text{if } dL_t > 0 \\ -\delta dt - dk_t^* & \text{if } dL_t = 0 \end{cases}$$

Let $f(s,t)$ denotes the density function for $s_{i,t}$. Since there is no aggregate shock in the economy, the cross-sectional density has settled into a steady state. Since the number of firms is large, the steady state cross-sectional density corresponds to the ergodic density of a single s_i. As each s_i behaves as a Brownian motion regulated at 0, with standard deviation σ and drift $\nu \equiv -(\mu - \sigma^2/2 + \delta)$, the steady state density is exponential (see appendix):

$$f(s) = \zeta e^{-\zeta s} \quad s \geq 0, \quad \text{where } \zeta \equiv -\frac{2\nu}{\sigma^2} \tag{50}$$

Figure 12 plots the steady state densities for two positive values of σ. With positive depreciation $\delta > 0$ and a secular tendency for desired investment to be positive ($\mu - \sigma^2/2 > 0$), we have $\zeta > 0$ and every individual tends to drift towards the investment point, where $s_t = 0$. Hence, in steady state more units are found in the neighborhood of $s = 0$ than farther from it. Because of the presence of the irreversibility constraint, the high volatility of productivity shocks makes firm’s investment risky and therefore reduce the incentive to invest. The larger is the volatility of shocks, the smaller is the measure of units investing at any point in time and thus the smoother is the slope of the cross-sectional density.

Figure 12: **The Cross Sectional Density of Firm Investment Rate**



This figures plots the cross sectional density for firm investment. The pink dashed line corresponds to smaller idiosyncratic productivity shock $\sigma = 0.2$, while the blue dotted line corresponds to larger idiosyncratic productivity shock $\sigma = 0.4$. As the volatility of idiosyncratic productivity shock becomes larger, more firms are constrained in the inaction region. Therefore, the cross sectional density is more spread out.

9.3 Stationary Distribution

Let $f(s,t)$ denotes the probability density of the process s_t with stochastic differential equation

$$ds(t) = \nu dt + \sigma dW(t), \quad \sigma > 0$$

where $\{W(t)\}$ is a standard Wiener process, and let $\{s\}$ be reflected at 0 and $\bar{s} > 0$. The function $f(s,t)$ can be derived by solving the forward Kolmogorov equation

$$\partial_t f(s,t) = \frac{1}{2}\sigma^2 \partial_{ss} f(s,t) - \nu \partial_s f(s,t) \tag{51}$$

with boundary conditions

$$\begin{aligned}\frac{1}{2}\sigma^2 \partial_s f(0,t) &= \nu f(0,t), \forall t \\ \frac{1}{2}\sigma^2 \partial_s f(s,t) &= \nu f(s,t), \forall t\end{aligned}$$

and given initial condition

$$f(s,0) = \bar{g}(s), \quad \int_0^S \bar{g}(s) ds = 1$$

Separating the variables, we write $f(s,t) = g(s)h(t)$ and obtain a couple of ordinary differential equations. In the t direction,

$$h'(t) + \lambda h(t) = 0$$

has general solution $h(t) = Ae^{-\lambda t}$, A is a constant of integration. In the s direction,

$$\begin{aligned}g''(s) + \zeta g'(s) - \lambda \frac{\zeta}{\nu} g(s) = 0 \\ g'(0) = -\zeta g(0) \\ g'(S) = -\zeta g(S)\end{aligned}$$

where $\zeta = -2\nu/\sigma^2, \zeta > 0$. It defines a Sturn-Liouville problem with characteristic equation

$$a^2 + \zeta a - \frac{\lambda}{\nu}\zeta = 0$$

If $\lambda \leq -\zeta\nu/4 = \zeta^2\sigma^2/8$, the roots are real and solutions taken the general form

$$g(s) = A_1 e^{a_1 s} + A_2 e^{a_2 s} \tag{52}$$

Solutions in this form need be considered only if they can satisfy the boundary condition with A_1 and/or A_2 different from 0. There exist solutions if $\lambda = 0$, which corresponding to the steady state equilibrium.

$$g(s) = \zeta e^{-\zeta s} \tag{53}$$